GABOR FRAMES BY SAMPLING AND PERIODIZATION

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Abstract. By sampling the window of a Gabor frame for $L^2(\mathbb{R})$ belonging to Feichtinger's algebra, $S_0(\mathbb{R})$, one obtains a Gabor frame for $l^2(\mathbb{Z})$. In this article we present a survey of results by R. Orr and A.J.E.M. Janssen and extend their ideas to cover interrelations among Gabor frames for the four spaces $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$, $L^2([0,L])$ and $\mathbb{C}^L$. Some new results about general dual windows with respect to sampling and periodization are presented as well. This theory is used to show a new result of the Kaiblinger type to construct an approximation to the canonical dual window of a Gabor frame for $L^2(\mathbb{R})$.

1. Introduction

To compute an approximation to the Fourier transform of a function supported on an interval, a common technique is to sample the function regularly and compute the Discrete Fourier Transform of the obtained sequence. The result is an approximation to the Fourier transform of the original function computed at regular sampled points.

This relation was first transferred to the case of Gabor systems by Orr in [21]. He shows how the discrete Gabor coefficients of a sampled and periodized function can be calculated from a periodization in both time and frequency of their continuous counterparts. Orr only considers critically sampled Gabor systems.

In [17] Janssen shows that under certain conditions one can obtain a Gabor frame for $l^2(\mathbb{Z})$ by sampling the window function of a Gabor frame for $L^2(\mathbb{R})$ at the integers. Furthermore, it is shown that the canonical dual windows of the two Gabor frames are also related by sampling. By using the same techniques, he shows how to produce Gabor frames for $\mathbb{C}^L$ by periodizing the window function of a Gabor frame for $l^2(\mathbb{Z})$.

The results by Orr and Janssen covers the transition from the continuous setting to the discrete setting. Based on these results, Kaiblinger [19] has constructed a method to go backwards, that is to construct a continuous approximation to the canonical dual window of a Gabor frame for $L^2(\mathbb{R})$ from computations done purely in the discrete setting. He uses splines to reconstruct the canonical dual window from samples computed in the finite discrete setting.

The purpose of this article is to extend the results of Orr and Janssen to all of the four spaces $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$, $L^2([0,L])$ and $\mathbb{C}^L$. We also provide new results about non-canonical dual windows. The situation is shown in Figure 1.1: We use these results to construct a new method for computing approximations to the canonical dual window of a Gabor frame for $L^2(\mathbb{R})$. The method uses the same basic approach as that of Kaiblinger.

$$\begin{tikzcd} L^2(\mathbb{R}) \arrow[r, "sampling"] \arrow[d, "periodization"] & l^2(\mathbb{Z}) \arrow[d] \\ L^2([0,L]) \arrow[r] & \mathbb{C}^L \end{tikzcd}$$

FIGURE 1.1. Relationship among the four different spaces for which we consider Gabor frames. Arrows to the right indicate sampling and arrows down indicate periodization.

In Section 2 the required background and notation is presented. It includes the definitions of the various operators used, commutation relations between the operators, window classes for the Gabor frames, and definitions of frames and Gabor frames for each of the four spaces.

Section 3 presents a survey of how and under which conditions Gabor frames for the four spaces $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$, $L^2([0,L])$ and $\mathbb{C}^L$ are interrelated by sampling and periodization operations.

For each relation in Figure 1.1, three types of results are presented:

(1) If the window function of a Gabor frame belongs to S_0, then a sampling/periodization of the window generates a Gabor frame with the same parameters and frame bounds for one of the other spaces. The canonical dual windows of the two Gabor frames are similarly related by sampling/periodization. These are simple extensions of the results in [17]
(2) For functions $f \in S_0$, simple relations hold between the expansion coefficients of f in a Gabor frame, and the expansion coefficients of the sampled/periodized function in the sampled/periodized Gabor frame. These are results of Orr from [21] made rigorous by Kaiblinger in [19].
(3) If $f, \gamma \in S_0$ are dual windows of a Gabor frame, the sampled/periodized windows are also dual windows of the sampled/periodized Gabor frame. These are new results.

Proof of the statements in this section will be postponed (unless very short) to the appendix.

In the last section, Section 4, the new method to "go backwards" is presented. We show how to use the elements of a Gabor frame to construct an approximation to the canonical dual window of this frame.

2. BASIC THEORY

The four spaces used in this article, $L^2(\mathbb{R})$,$L^2(\mathbb{Z})$,$L^2([0,L])$ and $\mathbb{C}^L$ share two common properties:

(1) They are Hilbert spaces. Therefore common results from frame theory apply to them.
(2) The domains $\mathbb{R}, \mathbb{Z}, [0,L]$ and $\mathbb{C}$ are locally compact Abelian (LCA) groups (the interval $[0,L]$ will always be thought of as a parameterization of the torus, $\mathbb{T}$). Most of the theory used in this article can be defined solely in terms of LCA-groups. This includes the Fourier

transform, the translation and modulation operators, the sampling and periodization operators, Gabor systems and the space S_0. We will not use the LCA-definitions, but instead define everything for each of the four spaces.

2.1. **The Fourier transforms.** The proofs in this article will make frequent use of various types of Fourier transformations. They are defined by

$$\begin{aligned}
\mathcal{F}_{\mathbb{R}} :\, \left(L^1 \cap L^2\right)(\mathbb{R}) \to L^2(\mathbb{R}) \quad &: \quad \left(\mathcal{F}_{\mathbb{R}} f\right)(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i \omega x} dx \\
\mathcal{F}_{\mathbb{Z}} : l^2(\mathbb{Z}) \to L^2([0,1]) \quad &: \quad \left(\mathcal{F}_{\mathbb{Z}} f\right)(\omega) = \sum_{k\in\mathbb{Z}} f(k) e^{-2\pi i k x} \\
\mathcal{F}_{[0,L]} :\, L^2([0,L]) \to l^2(\mathbb{Z}) \quad &: \quad \left(\mathcal{F}_{[0;L]} f\right)(k) = \frac{1}{\sqrt{L}} \int_0^L f(x) e^{-2\pi i k x/L} dx \\
\mathcal{F}_{L} :\, \mathbb{C}^L \to \mathbb{C}^L \quad &: \quad \left(\mathcal{F}_{L} f\right)(k) = \frac{1}{\sqrt{L}} \sum_{j=0}^{L-1} f(j) e^{-2\pi i k j/L}.
\end{aligned}$$

$\mathcal{F}_{\mathbb{R}}$ is the Fourier transform, $\mathcal{F}_{\mathbb{Z}} c$ is known as the frequency domain representation of c, $\mathcal{F}_{[0,L]} f$ is the Fourier coefficients of f, and $\mathcal{F}_L$ is the Discrete Fourier Transform. The notation is heavy, but it helps to avoid confusion when several different transforms are involved.

The Fourier transform $\mathcal{F}_{\mathbb{R}}$ can be extended to a bounded, linear, unitary operator, $\mathcal{F}_{\mathbb{R}} :\, L^2(\mathbb{R}) \to L^2(\mathbb{R})$. With this extension, all the transforms are unitary operators.

For brevity, we shall use the notation $\mathcal{F}$ and $\hat{f}$ for $\mathcal{F}_{\mathbb{R}}$ and $\mathcal{F}_{\mathbb{R}} f$ *only*! We shall always write the other types of Fourier transformation with a subscript.

2.2. **The translation, modulation and dilation operators.** Gabor frames for the four spaces are defined in terms of the translation and modulation operators on the spaces.

Definition 1. The translation operators:

$$\begin{aligned}
T_t :\, L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad &: \quad \left(T_t f\right)(x) = f(x-t),\ x,t \in \mathbb{R} \\
T_j :\, l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \quad &: \quad \left(T_j f\right)(k) = f(k-j),\ k,j \in \mathbb{Z} \\
T_t :\, L^2([0,L]) \to L^2([0,L]) \quad &: \quad \left(T_t f\right)(x) = f(x-t),\ x,t \in [0,L] \\
T_j :\, \mathbb{C}^L \to \mathbb{C}^L \quad &: \quad \left(T_j f\right)(k) = f(k-j),\ k,j \in \{0,\ldots,L-1\}.
\end{aligned}$$

The space $L^2([0,L])$ will always be thought of as a space of periodic functions, such that if $f \in L^2([0,L])$ then $f(x) = f(x+nL)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Definition 2. The modulation operators:

$$\begin{aligned}
M_\omega :\, L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad &: \quad \left(M_\omega f\right)(x) = e^{2\pi i \omega x} f(x),\ x,\omega \in \mathbb{R} \\
M_\omega :\, l^2(\mathbb{Z}) \to l^2(\mathbb{Z}) \quad &: \quad \left(M_\omega f\right)(j) = e^{2\pi i \omega j} f(j),\ j \in \mathbb{Z}, \omega \in [0,1] \\
M_k :\, L^2([0,L]) \to L^2([0,L]) \quad &: \quad \left(M_k f\right)(x) = e^{2\pi i k x/L} f(x),\ x \in [0,L],\ k \in \mathbb{Z} \\
M_k :\, \mathbb{C}^L \to \mathbb{C}^L \quad &: \quad \left(M_k f\right)(j) = e^{2\pi i k j/L} f(j),\ j,k \in \{0,\ldots,L-1\}.
\end{aligned}$$

The space $L^2([0,L])$ only permit modulations with integer parameters, because the exponential factor must have period L. The modulation operator for $l^2(\mathbb{Z})$ is periodic in its parameter because $e^{2\pi i\omega j} = e^{2\pi i(\omega+n)j}$ for all integers j and n.

Definition 3. Let $d > 0$. The *dilation* operators

$$D_d : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \quad : \quad (D_d f)(x) = \sqrt{d} f(dx), \quad \forall x \in \mathbb{R}.$$
$$D_d : L^2([0,L]) \to L^2([0,\frac{L}{d}]) \quad : \quad (D_d g)(x) = \sqrt{d} g(dx), \quad x \in [0,\frac{L}{d}].$$

2.3. **Window classes.** As a convenient window class for the four different types of Gabor frames we shall use the spaces $S_0(\mathcal{G})$, known as *Feichtinger's algebra,* which can be defined for LCA-groups, [6]. For the LCA-groups considered in this article these are the spaces $S_0(\mathbb{R})$, $l^1(\mathbb{Z})$, $\mathcal{A}([0,L])$ and $\mathbb{C}^L$. This gives window classes for $L^2(\mathbb{R})$, $l^2(\mathbb{Z})$, $L^2([0,L])$ and $\mathbb{C}^L$respectively.

Definition 4. A function $g \in L^2(\mathbb{R})$ belongs to *Feichtinger's algebra* $S_0(\mathbb{R})$ if

$$\|g\|_{S_0} = \int_{\mathbb{R}\times\mathbb{R}} \left| \int_{\mathbb{R}} g(t)\overline{(M_\omega T_x \varphi)(t)} dt \right| dx d\omega < \infty,$$

where $\varphi(t) = e^{-\pi t^2}$ is the Gaussian function.

Replacing the Gaussian φ in the definition of S_0 with another non-zero window in the Schwartz class of smooth, exponentially decaying functions, will give an equivalent norm. The space $S_0(\mathbb{R})$ is invariant under translation, modulation, dilation and the Fourier transform. Furthermore, $S_0(\mathbb{R}) \subseteq L^p(\mathbb{R})$ for $1 \le p \le \infty$. Functions in $S_0(\mathbb{R})$ are continuous. These and other useful properties of $S_0(\mathbb{R})$ can be found in [11] and [12]. There exists many useful conditions for membership of $S_0(\mathbb{R})$, see [11, 12, 20].

For Gabor frames on the interval $[0,L]$ we shall use the window class $S_0([0,L]) = \mathcal{A}([0,L]) = \mathcal{F}^{-1}_{[0,L]} l^1(\mathbb{Z})$. These are the functions on the interval $[0,L]$ having an absolutely convergent Fourier series. $\mathcal{A}([0,L])$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{A}([0,L])} \quad = \quad \left\| \mathcal{F}^{-1}_{[0,L]} f \right\|_{l^1(\mathbb{Z})},$$

and is a Banach algebra with respect to point-wise multiplication. If $g \in \mathcal{A}([0,L])$ then g is a continuous function.

The Gabor coefficients of an S_0-function with respect to an S_0-window belongs to l^1. The precise relations are as follows, see [6, 11] and [12, Cor. 12.1.12].

Proposition 5. *Summability of Gabor coefficients.*

$S_0(\mathbb{R})$ *Let* $g, \gamma \in S_0(\mathbb{R})$ *and* $\alpha, \beta > 0$. *Then*

$$\sum_{m,n\in\mathbb{Z}} |\langle \gamma, M_{m\beta} T_{n\alpha} g\rangle| < \infty$$

$l^1(\mathbb{Z})$ *Let* $g, \gamma \in l^1(\mathbb{Z})$ *and* $a, M \in \mathbb{N}$. *Then*

$$\sum_{m=0}^{M-1} \sum_{n\in\mathbb{Z}} \left|\langle \gamma, M_{m/M} T_{na} g\rangle\right| < \infty$$

$\mathcal{A}([0,L])$ *Let $g,\gamma \in \mathcal{A}([0,L])$ and $N, b \in \mathbb{N}$ with $L = Na$. Then*

$$\sum_{m\in\mathbb{Z}} \sum_{n=0}^{N-1} |\langle \gamma, M_{mb}T_{na}g\rangle| < \infty$$

2.4. **The sampling and periodization operators.** The process of sampling is well-defined on $S_0(\mathbb{R})$:

Definition 6. Let $\alpha > 0$ and $a \in \mathbb{N}$. The sampling operators are given by

$$\begin{aligned}
\mathcal{S}_\alpha : S_0(\mathbb{R}) \to l^1(\mathbb{Z}) \quad &: \quad (\mathcal{S}_\alpha f)(j) = \sqrt{\alpha} f(j\alpha), \quad \forall j \in \mathbb{Z} \\
\mathcal{S}_a : l^1(\mathbb{Z}) \to l^1(\mathbb{Z}) \quad &: \quad (\mathcal{S}_a f)(j) = \sqrt{a} f(ja), \quad \forall j \in \mathbb{Z} \\
\mathcal{S}_\alpha : C([0;\alpha L]) \to \mathbb{C}^L \quad &: \quad (\mathcal{S}_\alpha f)(j) = \sqrt{\alpha} f(j\alpha), \quad j = 0,...,L-1 \\
\mathcal{S}_a : \mathbb{C}^{aL} \to \mathbb{C}^L \quad &: \quad (\mathcal{S}_a f)(j) = \sqrt{a} f(ja), \quad j = 0,...,L-1
\end{aligned}$$

The fact that the sampling operator is a bounded operator from $S_0(\mathbb{R})$ into $l^1(\mathbb{Z})$ is proved in [11, Lemma 3.2.11]. The factor $\sqrt{\alpha}$ appearing in the definition of the sampling operator gives the sampling operators the following important properties:

- Composition with dilations:

$$\mathcal{S}_a D_b = \mathcal{S}_{ab} \text{ on } S_0(\mathbb{R}) \tag{2.1}$$

$$\mathcal{S}_a D_b = \mathcal{S}_{ab} \text{ on } C([0,L]) \tag{2.2}$$

- If $f \in S_0(\mathbb{R})$ then $\|S_\alpha f\| < C \|f\|_{S_0}$ for some $C > 0$ independent of α. This is proved in [12, Prop. 11.1.4].

Definition 7. The periodization operators are given by

$$\begin{aligned}
\mathcal{P}_L : S_0(\mathbb{R}) \to \mathcal{A}([0,L]) \quad &: \quad \mathcal{P}_L g(x) = \sum_{k\in\mathbb{Z}} g(x+kL), \quad x \in [0,L] \\
\mathcal{P}_L : l^1(\mathbb{Z}) \to \mathbb{C}^L \quad &: \quad \mathcal{P}_L g(j) = \sum_{k\in\mathbb{Z}} g(j+kL), \quad j = 0,...,L-1 \\
\mathcal{P}_M : \mathcal{A}([0,ML]) \to \mathcal{A}([0,L]) \quad &: \quad \mathcal{P}_M g(x) = \sum_{k\in\mathbb{Z}} g(x+kM), \quad x \in [0,L] \\
\mathcal{P}_M : \mathbb{C}^{ML} \to \mathbb{C}^L \quad &: \quad \mathcal{P}_M g(j) = \sum_{k\in\mathbb{Z}} g(j+kM), \quad j = 0,...,L-1
\end{aligned}$$

The fact that $\mathcal{P}_L : S_0(\mathbb{R}) \to \mathcal{A}([0,L])$ is proved in [8] in the more general context of LCA-groups.

2.5. **Frames for Hilbert spaces.** Since we will deal with Gabor frames for four different spaces, it will be beneficial to note what can be said about frames for general Hilbert spaces.

Definition 8. A family of elements $\{e_j\}_{j\in J}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if constants $0 < A \leq B < \infty$ exist such that

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{j\in J} \left|\langle f, e_j\rangle_{\mathcal{H}}\right|^2 \leq B\|f\|_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}. \tag{2.3}$$

The constants A and B are called lower and upper frame bounds, respectively.

Two frames for a Hilbert space $\mathcal{H}$, $\{e_j\}$ and $\{f_j\}$, are called *dual frames* if and only if $f = \sum_{j\in J} \langle f, e_j\rangle f_j, \quad \forall f \in \mathcal{H}$.

The *frame operator* of a frame $\{e_j\}_{j\in J}$ for a Hilbert space $\mathcal{H}$ is defined by

$$S : \mathcal{H} \to \mathcal{H} \quad : \quad Sf = \sum_j \langle f, e_j\rangle_{\mathcal{H}}\, e_j, \tag{2.4}$$

where the series defining Sf converges unconditionally for all $f \in \mathcal{H}$.

The condition (2.3) ensures that the frame operator is both bounded and invertible on $\mathcal{H}$.

The inverse frame operator can be used to give a decomposition of any function $f \in \mathcal{H}$:

$$f = \sum_j \left\langle f, S^{-1}e_j\right\rangle e_j, \quad \forall f \in \mathcal{H}. \tag{2.5}$$

The frame $\left\{S^{-1}e_j\right\}$ is known as the canonical dual frame.

2.6. **Gabor systems.** Gabor systems can be defined for the four spaces using the corresponding translation and modulation operators.

Definition 9. Gabor systems for each of the four spaces:

$L^2(\mathbb{R})$ A Gabor system (g, α, β) for $L^2(\mathbb{R})$ is defined as

$$(g, \alpha, \beta) = \{M_{m\beta} T_{n\alpha} g\}_{m,n\in\mathbb{Z}}\,, \tag{2.6}$$

where $g \in L^2(\mathbb{R})$ and $\alpha, \beta > 0$.

$l^2(\mathbb{Z})$ A Gabor system $\left(g, a, \frac{1}{M}\right)$ for $l^2(\mathbb{Z})$ is defined as

$$\left(g, a, \frac{1}{M}\right) = \left\{M_{m/M} T_{na} g\right\}_{m=0,\dots,M-1,n\in\mathbb{Z}}\,, \tag{2.7}$$

where $g \in l^2(\mathbb{Z})$ and $a, M \in \mathbb{N}$.

$L^2([0, L])$ A Gabor system (g, a, b) for $L^2([0, L])$ is defined as

$$(g, a, b) = \{M_{mb} T_{na} g\}_{m\in\mathbb{Z},n=0,\dots,N-1}\,, \tag{2.8}$$

where $g \in L^2([0, L])$, $a, b, N \in \mathbb{N}$ and $L = Na$.

$\mathbb{C}^L$ A Gabor system for (g, a, b) for $\mathbb{C}^L$ is defined as

$$(g, a, b) = \{M_{mb} T_{na} g\}_{m=0,\dots,M-1,n=0,\dots,N-1}\,, \tag{2.9}$$

where $g \in \mathbb{C}^L$, $a, b, M, N \in \mathbb{N}$ and $Mb = Na = L$.

A *Gabor frame* is a Gabor system that is also a frame. For a good introduction to Gabor systems see [9, 10] and [12].

The Gabor frame operator (and its inverse) commute with the translation and modulation operators. If this is applied to the the decomposition formula (2.5), it shows that the canonical dual frame of a Gabor frame (g, α, β) is again a Gabor frame $\left(\gamma^0, \alpha, \beta\right)$, where $\gamma^0 = S^{-1}g$ is known as the *canonical dual window.*

If a Gabor frame is not a Riesz basis, it has more than one dual frame, and some of these dual frames will even be Gabor frames.

3. Between the spaces

The section is a survey of existing results of how Gabor frames behave with respect to samplings and periodization. Some new results about non-canonical dual window for Gabor frames are presented as well.

We present the results for the interrelations show on Fig. 1.1. This reason for this is that the four different types of Gabor systems each has their place in applications. Gabor frames for $l^2(\mathbb{Z})$ and $L^2([0, L])$ are less common in literature, but there are several good reasons for considering them:

- The space $l^2(\mathbb{Z})$ is a good model of a possibly infinite stream of discrete data, as it occurs in filter bank theory. It is also the correct place to study decay properties of discrete Gabor windows. Results about decay properties of windows for Gabor frames for $l^2(\mathbb{Z})$ can be transferred to the finite, discrete case by the results presented in Sec. 3.4.
- The space $L^2([0, L])$ is a good model of continuous phenomena with finite duration. An advantage of using $L^2([0, L])$ as opposed to $L^2(\mathbb{R})$ is that discretizing a Gabor frame is straightforward, see Sec. 3.3.

A generalization that will not be explicitly mentioned in the following is, that all statements about a canonical dual window also holds for the corresponding tight window. Proofs of this appear in [4].

3.1. **From $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$.** In [17] Janssen proved, that if the window g of a Gabor frame $\left(g, a, \frac{1}{M}\right)$ for $L^2(\mathbb{R})$, $a, M \in \mathbb{N}$, satisfies the so-called *condition R* then by sampling it at the integers one obtains the window $\mathcal{S}_1 g$ of a Gabor frame $\left(\mathcal{S}_1 g, a, \frac{1}{M}\right)$ for $l^2(\mathbb{Z})$ with the same frame bounds. Furthermore he proved that if g additionally satisfies the so-called *condition A* then $\mathcal{S}_1 \gamma^0$ will be the canonical dual window of $\left(\mathcal{S}_1 g, a, \frac{1}{M}\right)$.

Condition R is a regularity condition, that requires the function to have some smoothness around sampling points and decay as well. It is much weaker than requiring the function to be in $S_0(\mathbb{R})$. Most remarkably, it only places restrictions on g in a neighbourhood of the sampling points. The downside of condition R is that it depends on the position of the sampling points.

Condition A is a statement about the decay of inner products between g and certain time-frequency shifts of g. It depends on the parameters α, β, and is in general hard to verify.

All functions in $S_0(\mathbb{R})$ satisfies condition R and A for all sampling distances and for any combination of α and β. For condition R this is mentioned in [4]. For condition A this is a simple fact of Proposition 5. A direct proof is in [12, Corr. 12.1.12], along with a nice coverage of the topic.

In [19], new proofs of the combined transition $L^2(\mathbb{R}) \to \mathbb{C}^L$ is presented. These use $S_0(\mathbb{R})$ as a sufficient condition, and does not consider condition R and A.

The following result is a simple generalization of Janssen's sampling result to hold for Gabor frames with rational oversampling.

Theorem 10. *Let $g \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{a}{M}$ for some $M, a \in \mathbb{N}$, and assume that (g, α, β) is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual* *window* γ^0*. Then* $\left(\mathcal{S}_{\alpha/a}g, a, \frac{1}{M}\right)$ *is a Gabor frame for* $l^2(\mathbb{Z})$ *with the same frame bounds and canonical dual window* $\mathcal{S}_{\alpha/a}\gamma^0$*.*

In [21] Orr showed that the expansion coefficients $c(m,n)$ of a function f in a Gabor frame for $L^2(\mathbb{R})$ with dual window γ is related to the expansion coefficients $d(m,n)$ of the sampled function $\mathcal{S}_{\alpha/a}f$ in a Gabor frame for $l^2(\mathbb{Z})$ with the sampled dual window $\mathcal{S}_{\alpha/a}\gamma$. In [21] the result is stated without explicit conditions on the involved functions and only in the case of critical sampling (when $\alpha\beta = 1$). In [19] Kaiblinger has shown that $S_0(\mathbb{R})$ is a sufficient condition, and that the statements also holds for oversampled Gabor frames. The result [19, Lem. 9] covers the complete transition $L^2(\mathbb{R}) \to \mathbb{C}^L$. The proof of the following statement can be found by modifying this.

Proposition 11. *Let* $f, \gamma \in S_0(\mathbb{R})$*,* $\alpha, \beta > 0$ *with* $\alpha\beta = \frac{a}{M}$ *for some* $M, a \in \mathbb{N}$*. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{m\beta}T_{n\alpha}\gamma\rangle_{L^2(\mathbb{R})}, \quad m,n \in \mathbb{Z}, \\ d(m,n) &= \left\langle \mathcal{S}_{\alpha/a}f, M_{m/M}T_{na}\mathcal{S}_{\alpha/a}\gamma\right\rangle_{l^2(\mathbb{Z})}, \quad m = 0,...,M-1, n \in \mathbb{Z} \end{aligned}$$

then

$$d(m,n) = \sum_{j\in\mathbb{Z}} c(m - jM, n), \quad \forall m = 0,...,M-1, n \in \mathbb{Z}$$

The sum in the previous proposition is well defined because of Prop. 5.

Theorem 10 shows that if γ is the canonical dual window of g, $\mathcal{S}_{\alpha/a}\gamma$ is the canonical dual window of $\mathcal{S}_{\alpha/a}g$. This relation holds not only for the canonical dual window, but in fact for ALL dual windows $\gamma \in S_0(\mathbb{R})$.

Proposition 12. *Let* (g, α, β)*,* $g \in S_0(\mathbb{R})$*,* $\alpha, \beta > 0$ *with* $\alpha\beta = \frac{a}{M}$ *for some* $M, a \in \mathbb{N}$*, be a Gabor frame for* $L^2(\mathbb{R})$ *and let* $\gamma \in S_0(\mathbb{R})$ *be a dual window. Then* $S_{\alpha/a}\gamma$ *is a dual window of* $\left(\mathcal{S}_{\alpha/a}g, a, \frac{1}{M}\right)$*.*

In [16] it is shown that if g satisfies Condition A then the canonical dual window γ^0 of (g, α, β) also satisfies Condition A. In [7, 13] it is shown that if $g \in S_0(\mathbb{R})$ then also $\gamma^0 \in S_0(\mathbb{R})$. Because of these results, the only assumption needed in Theorem 10 is $g \in S_0(\mathbb{R})$. In Proposition 12 we need to impose $\gamma^0 \in S_0(\mathbb{R})$ as a separate condition.

3.2. **From $L^2(\mathbb{R})$ to $L^2([0, L])$.** The results from the previous section can be easily extended to prove how a Gabor frame for $L^2([0, L])$ can be obtained from a Gabor frame for $L^2(\mathbb{R})$ by periodizing the window function.

Gabor frames for $L^2([0, L])$ are not as widely studied as Gabor frames for $L^2(\mathbb{R})$. From a practical point of view, they are interesting because they can be used to model continuous phenomena of finite duration.

Interrelations between Wilson bases for $L^2(\mathbb{R})$ and $L^2([0, L])$ are described in [1, Corollary 9.3.6].

Theorem 13. *Let* $g \in S_0(\mathbb{R})$*,* $\alpha, \beta > 0$ *with* $\alpha\beta = \frac{b}{N}$ *for some* $N, b \in \mathbb{N}$ *and assume that* (g, α, β) *is Gabor frame for* $L^2(\mathbb{R})$ *with canonical dual window* γ^0*. Then* $\left(\mathcal{P}_{b/\beta}g, \alpha, b\right)$ *is a Gabor frame for* $L^2([0, \frac{b}{\beta}])$ *with the same frame bounds and canonical dual window* $\mathcal{P}_{b/\beta}\gamma^0$*.*

The proof of the following can by modifying [19, Lem. 9].

Proposition 14. *Let $f, \gamma \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{b}{N}$ for some $N, b \in \mathbb{N}$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{m\beta} T_{n\alpha} \gamma \rangle_{L^2(\mathbb{R})}, \quad m, n \in \mathbb{Z} \\ d(m,n) &= \left\langle \mathcal{P}_{b/\beta} f, M_{mb} T_{na} \mathcal{P}_{b/\beta} \gamma \right\rangle_{L^2([0, \frac{b}{\beta}])}, \quad m \in \mathbb{Z}, n = 0, ..., N-1 \end{aligned}$$

then

$$d(m,n) = \sum_{j \in \mathbb{Z}} c(m, n - jN), \quad \forall m \in \mathbb{Z}, n = 0, ..., N-1.$$

The sum in the previous proposition is well defined because of Prop. 5.

Proposition 15. *Let (g, α, β) with $g \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ and $\alpha\beta = \frac{b}{N}$ for some $N, b \in \mathbb{N}$ be a Gabor frame for $L^2(\mathbb{R})$ and let $\gamma \in S_0(\mathbb{R})$ be a dual window. Then $\mathcal{P}_{b/\beta}\gamma$ is a dual window of $\left(\mathcal{P}_{b/\beta} g, \alpha, b\right)$.*

Proof. The proof has the same structure as that of Proposition 12, using Proposition 14 as the main ingredient. □

3.3. **From $L^2([0, L])$ to $\mathbb{C}^L$.** A Gabor frame for $\mathbb{C}^L$ can be obtained by sampling the window function of a Gabor frame for $L^2([0, L])$. The proofs are very similar to the proofs presented in Section 3.1 for the $L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ case.

Theorem 16. *Let $g \in \mathcal{A}([0, L_1])$, $L_2, M, N, a_2, b \in \mathbb{N}$ with $L_1 = Na_1$ and $L_2 = Mb = Na_2$. Assume that (g, a_1, b) is a Gabor frame for $L^2([0, L_1])$ with canonical dual window γ^0. Then $\left(\mathcal{S}_{L_1/L_2} g, a_2, b\right)$ is a Gabor frame for $\mathbb{C}^{L_2}$ with the same frame bounds and canonical dual window $\mathcal{S}_{L_1/L_2}\gamma^0$.*

Proof. The proof can be found by carefully modifying the proof in [17] for the $L^2(\mathbb{R})$-case. The modifications consist in replacing integration over $\mathbb{R}$ with integrations over $[0, L]$ and similar changes. □

The proof of the following can by modifying [19, Lem. 9].

Proposition 17. *Let $f, \gamma \in \mathcal{A}([0, L_1])$, $L_2, M, N, a_2, b \in \mathbb{N}$ with $L_1 = Na_1$ and $L_2 = Mb = Na_2$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{mb} T_{na_1} \gamma \rangle_{L^2([0, L_1])}, \quad m \in \mathbb{Z}, n = 0, ..., N-1 \\ d(m,n) &= \left\langle \mathcal{S}_{L_1/L_2} f, M_{mb} T_{na_2} \mathcal{S}_{L_1/L_2} \gamma \right\rangle_{\mathbb{C}^L}, \ m = 0, ..., M-1, n = 0, ..., N-1 \end{aligned}$$

then

$$d(m,n) = \sum_{j \in \mathbb{Z}} c(m - jM, n), \quad \forall m = 0, ..., M-1, n = 0, ..., N-1$$

The sum in the previous proposition is well defined because of Prop. 5.

Proposition 18. *Let (g, a_1, b) $g \in \mathcal{A}([0, L_1])$, $L_2, M, N, a_2, b \in \mathbb{N}$ with $L_1 = Na_1$ and $L_2 = Mb = Na_2$, be a Gabor frame for $L^2([0, L_1])$ and let $\gamma \in \mathcal{A}([0, L_1])$ be a dual window. Then $\mathcal{S}_{L_1/L_2}\gamma$ is a dual window of $\left(\mathcal{S}_{L_1/L_2} g, a_2, b\right)$.*

Proof. The proof has the same structure as that of Proposition 12, using Proposition 17 as the main ingredient. □

3.4. **From $l^2(\mathbb{Z})$ to $\mathbb{C}^L$.** Similarly to the results presented in the previous sections, one can obtain a Gabor frame for $\mathbb{C}^L$ by periodizing the window function of a Gabor frame for $l^2(\mathbb{Z})$.

The following result appears in[17].

Theorem 19. *Let $g \in l^1(\mathbb{Z})$, $M, N, a, b \in \mathbb{N}$ with $Mb = Na = L$ and assume that $\left(g, a, \frac{1}{M}\right)$ is a Gabor frame for $l^2(\mathbb{Z})$ with canonical dual window γ^0. Then $(\mathcal{P}_L g, a, b)$ is a Gabor frame for $\mathbb{C}^L$ with the same frame bounds and canonical dual window $\mathcal{P}_L \gamma^0$.*

The proof of the following can by modifying [19, Lem. 9].

Proposition 20. *Let $f, \gamma \in l^1(\mathbb{Z})$, $M, N, a, b \in \mathbb{N}$ with $Mb = Na = L$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{mb} T_{na} \gamma \rangle_{l^2(\mathbb{Z})}, \quad m = 0, ..., M-1, n \in \mathbb{Z} \\ d(m,n) &= \langle \mathcal{P}_L f, M_{mb} T_{na} \mathcal{P}_L \gamma \rangle_{\mathbb{C}^L}, \ m = 0, ..., M-1, n = 0, ..., N-1, \end{aligned}$$

then

$$d(m,n) = \sum_{j \in \mathbb{Z}} c(m, n - jN), \quad \forall m = 0, ..., M-1, n = 0, ..., N-1$$

The sum in the previous proposition is well defined because of Prop. 5.

Proposition 21. *Let $\left(g, a, \frac{1}{M}\right)$ $g \in l^1(\mathbb{Z})$, $M, N, a, b \in \mathbb{N}$ with $Mb = Na = L$, be a Gabor frame for $l^2(\mathbb{Z})$ and let $\gamma \in l^1(\mathbb{Z})$ be a dual window. Then $\mathcal{P}_L \gamma$ is a dual window of $(\mathcal{P}_L g, a, b)$.*

Proof. The proof has the same structure as that of Proposition 12, using Proposition 20 as the main ingredient. □

3.5. **From $L^2(\mathbb{R})$ to $\mathbb{C}^L$.** With the results from the previous sections, any Gabor frame for $L^2(\mathbb{R})$, (g, α, β) with $g \in S_0(\mathbb{R})$ and rational sampling, $\alpha\beta \in \mathbb{Q}$, can be sampled and periodized to obtain a Gabor frame for $\mathbb{C}^L$ with corresponding sampled and periodized canonical dual window. Direct proofs of the two first statements are given in [19]. The proofs of the three following statements can also found by simply combining the similar from the previous sections. Going through $L^2([0, \frac{b}{\beta}])$ or $l^2(\mathbb{Z})$ produces the same result, because $\mathcal{P}_L \mathcal{S}_{\alpha/a} = \mathcal{S}_{b/\beta/L} \mathcal{P}_{b/\beta}$ on $S_0(\mathbb{R})$. This shows that the two directions on Fig. 1.1 commute.

Theorem 22. *Let $g \in S_0(\mathbb{R})$, $\alpha\beta = \frac{a}{M} = \frac{b}{N}$ and $Mb = Na = L$ with $a, b, M, N, L \in \mathbb{N}$ and assume that (g, α, β) is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0. Then $\left(\mathcal{P}_L \mathcal{S}_{\alpha/a} g, a, b\right)$ is a Gabor frame for $\mathbb{C}^L$ with the same frame bounds and canonical dual window $\mathcal{P}_L \mathcal{S}_{\alpha/a} \gamma^0$.*

Fully written out then $\mathcal{P}_L \mathcal{S}_{\alpha/a} g$ is

$$\left(\mathcal{P}_L \mathcal{S}_{\alpha/a}\right) g(j) = \sqrt{\frac{\alpha}{a}} \sum_{k \in \mathbb{Z}} g\left(\frac{\alpha}{a}(j - kL)\right), \quad j = 0, ..., L-1.$$

Proposition 23. *Let $f, \gamma \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{a}{M} = \frac{b}{N}$, $L = Mb = Na$ with $a, b, M, N, L \in \mathbb{N}$. With*

$$\begin{aligned} c(m,n) &= \langle f, M_{m\beta} T_{n\alpha} \gamma \rangle_{L^2(\mathbb{R})}, \quad m,n \in \mathbb{Z} \\ d(m,n) &= \left\langle \mathcal{P}_L \mathcal{S}_{\alpha/a} f, M_{mb} T_{na} \mathcal{P}_L \mathcal{S}_{\alpha/a} \gamma \right\rangle_{\mathbb{C}^L}, \; m = 0, ..., M-1, n = 0, ..., N-1, \end{aligned}$$

then

$$d(m,n) = \sum_{j,k \in \mathbb{Z}} c(m - kM, n - jN), \quad \forall = 0, ..., M-1, n = 0, ..., N-1.$$

The sum in the previous proposition is well defined because of Prop. 5.

Proposition 24. *Let $(g, \alpha\beta)$, $\alpha\beta = \frac{a}{M} = \frac{b}{N}$ and $Mb = Na = L$ with $a, b, M, N, L \in \mathbb{N}$, be a Gabor frame for $L^2(\mathbb{R})$ and let $\gamma \in S_0(\mathbb{R})$ be a dual window. Then $\mathcal{P}_L \mathcal{S}_{\alpha/a} \gamma$ is a dual window of $\left(\mathcal{P}_L \mathcal{S}_{\alpha/a} g, a, b\right)$.*

4. Going back

Theorem Theorem 22 provides an easy way to compute samples of the canonical dual window γ^0 of a Gabor frame (g, α, β) for $L^2(\mathbb{R})$ using finite-dimensional methods. The question is whether it is possible to use this to construct a sequence of functions converging to γ^0 in some norm.

Kaiblinger showed in [19] that this is indeed possible. By using a standard interpolation scheme to interpolate the computed samples, one can construct functions that converges to γ^0 in the $S_0(\mathbb{R})$-norm. This implies convergence for all L^p-spaces, $1 \le p \le \infty$.

Some other methods of approximating the inverse frame operator of Gabor frame is presented in [2, 4].

The following method works the same way as the method proposed by Kaiblinger, but it uses another interpolation scheme. We will use the Gabor atoms of the Gabor frame (g, α, β).

The result is presented only for M, N being even. The case of odd M, N is similar.

Theorem 25. *Let $g \in S_0(\mathbb{R})$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{a}{M}$ and assume that (g, α, β) is a Gabor frame for $L^2(\mathbb{R})$ with canonical dual window γ^0.*

For each even $M, N \in \mathbb{N}$ such that $L = Mb = Na$ with $a, b, L \in \mathbb{N}$ denote the canonical dual window of $\left(\mathcal{P}_L \mathcal{S}_{\alpha/a} g, a, b\right)$ by $\varphi_{M,N}$. Define $d_{M,N} \in \mathbb{C}^{M \times N}$ by $d_{M,N}(m,n) = \langle \varphi_{M,N}, M_{mb} T_{na} \varphi_{M,N} \rangle_{\mathbb{C}^L}$ and $\gamma^0_{M,N} \in S_0(\mathbb{R})$ by

$$\gamma^0_{M,N} = \sum_{n=-N/2}^{N/2-1} \sum_{m=-M/2}^{M/2-1} d_{M,N}(m,n) M_{m\beta} T_{n\alpha} g. \tag{4.1}$$

Then $\left\| \gamma^0 - \gamma^0_{M,N} \right\|_{S_0} \to 0$ as $M, N \to \infty$.

Proof. Define $c \in l^1(\mathbb{Z} \times \mathbb{Z})$ by $c(m,n) = \left\langle \gamma^0, M_{m\beta} T_{n\alpha} \gamma^0 \right\rangle_{L^2(\mathbb{R})}$. The crucial fact that $c \in l^1(\mathbb{Z} \times \mathbb{Z})$ follows from Prop. 5. Then

$$\gamma^0 = \sum_{m,n \in \mathbb{Z}} c(m,n) M_{m\beta} T_{n\alpha} g. \tag{4.2}$$

This is the standard frame expansion, since γ^0 and g are dual windows.

By (4.1) and (4.2) both γ^0 and $\gamma^0_{M,N}$ can be written using the frame (g,α,β). Subtracting them gives

$$\gamma^0 - \gamma^0_{M,N} = \sum_{m,n\in\mathbb{Z}} r_{M,N}(m,n) M_{m\beta} T_{n\alpha} g,$$

where

$$r_{M,N}(m,n) = \begin{cases} c(m,n) - d_{M,N}(m,n) & \text{if } -\frac{M}{2} \le m \le \frac{M}{2}-1 \text{ and } -\frac{N}{2} \le n \le \frac{N}{2}-1 \\ c(m,n) & \text{otherwise} \end{cases}$$

By Proposition 23,

$$d_{M,N}(m,n) = \sum_{j,k\in\mathbb{Z}} c(m-kM, n-jN), \tag{4.3}$$

for all $-\frac{M}{2} \le m \le \frac{M}{2}-1$ and $-\frac{N}{2} \le n \le \frac{N}{2}-1$. This gives the following expression of the residual r:

$$r_{M,N}(m,n) = \\ = \begin{cases} -\sum_{j,k\in\mathbb{Z}\setminus\{0\}} c_{M,N}(m-kM, n-jN) & \text{if } -\frac{M}{2} \le m \le \frac{M}{2}-1 \text{ and } -\frac{N}{2} \le n \le \frac{N}{2}-1 \\ c(m,n) & \text{otherwise} \end{cases}$$

From 4 it can be seen that the translation and modulation operators are unitary on $S_0(\mathbb{R})$. Using this, we get the following estimate.

$$\begin{aligned} \left\|\gamma^0 - \gamma^0_{M,N}\right\|_{S_0} &= \left\| \sum_{m,n\in\mathbb{Z}} r_{M,N}(m,n) M_{m\beta} T_{n\alpha} g \right\|_{S_0} \\ &\le \|g\|_{S_0} \sum_{m,n\in\mathbb{Z}} |r_{M,N}(m,n)| \\ &\le \|g\|_{S_0} \sum_{n=-N/2}^{N/2-1} \sum_{m=-M/2}^{M/2-1} \sum_{j,k\in\mathbb{Z}\setminus\{0\}} |c_{M,N}(m-kM, n-jN)| + \\ &\quad + \|g\|_{S_0} \sum_{m\notin\left\{-\frac{M}{2},\dots,\frac{M}{2}-1\right\}} \sum_{n\notin\left\{-\frac{N}{2},\dots,\frac{N}{2}-1\right\}} |c(m,n)| \end{aligned}$$

In the last term, only the coefficients outside the rectangle indexed by $m \in \left\{-\frac{M}{2},\dots,\frac{M}{2}-1\right\}$ and $n \in \left\{-\frac{N}{2},\dots,\frac{N}{2}-1\right\}$ appears, and they all appear twice. From this

$$\left\|\gamma^0 - \gamma^0_{M,N}\right\|_{S_0} \le 2\|g\| \sum_{m\notin\left\{-\frac{M}{2},\dots,\frac{M}{2}-1\right\}} \sum_{n\notin\left\{-\frac{N}{2},\dots,\frac{N}{2}-1\right\}} |c(m,n)|$$

When $M,N\to\infty$, the last term goes to zero. $\square$

Remark 26. Since each $\gamma^0_{M,N}$ is a finite linear combination of Gabor atoms from (g, α, β), they inherit properties from g: Since $g \in S_0(\mathbb{R})$ then each $\gamma^0_{M,N} \in S_0(\mathbb{R})$. Similarly, if g or $\hat{g}$ has exponential decay, then so does $\gamma^0_{M,N}$ or $\hat{\gamma}^0_{M,N}$. This is a main difference to the method of Kaiblinger [19], where the smoothness properties of the constructed approximation depend on the interpolation method.

Remark 27. If additionally $g \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz-space of rapidly decaying, smooth functions, then also $\gamma^0 \in \mathcal{S}(\mathbb{R})$, [15]. Is this case then by Remark 26 each $\gamma^0_{M,N} \in \mathcal{S}(\mathbb{R})$, so the convergence of $\gamma^0_{M,N}$ to γ^0 is purely within $\mathcal{S}(\mathbb{R})$.

To use this method, the two main numerical calculations to be carried out are the inversion of the frame operator for $(g_{M,N}, a, b)$ and the calculation of the coefficients $d_{M,N} \in \mathbb{C}^{M\times N}$. Algorithms based on FFTs and matrix-factorisations can be found in [22]. These calculations can be performed in $\mathcal{O}(Lq) + \mathcal{O}(NM \log M)$ time, where $\frac{q}{p}$ is the oversampling factor written as an irreducible fraction.

Appendix A. Proofs

A.1. **Some additional theory.** The dilation operators are unitary operators for all $d > 0$, and the following commutation relations between the translation and modulation operators holds:

$$D_d M_\omega T_t = M_{d\omega} T_{\frac{t}{d}} D_d \text{ on } L^2(\mathbb{R}) \tag{A.1}$$

$$D_d M_k T_t = M_k T_{\frac{t}{d}} D_d \text{ on } L^2([0, L]) \tag{A.2}$$

Notice that the parameter of the modulation operator is unchanged in (A.2), as opposed to (A.1). This is caused by the definition of the modulation operator.

For the various Fourier transforms and the modulation and translation operators the following commutation relations hold:

$$\mathcal{F}_\mathbb{R} M_\omega T_t = e^{2\pi i t\omega} M_{-t} T_\omega \mathcal{F}_\mathbb{R} \text{ on } L^2(\mathbb{R}) \tag{A.3}$$

$$\mathcal{F}_\mathbb{Z} M_\omega T_j = e^{2\pi i t\omega} M_{-j} T_\omega \mathcal{F}_\mathbb{Z} \text{ on } l^2(\mathbb{Z}) \tag{A.4}$$

$$\mathcal{F}_{[0,L]} M_k T_t = e^{2\pi i t k/L} M_{-t/L} T_k \mathcal{F}_{[0,L]} \text{ on } L^2([0, L]) \tag{A.5}$$

$$\mathcal{F}_L M_k T_j = e^{2\pi i jk/L} M_{-j} T_k \mathcal{F}_L \text{ on } \mathbb{C}^L \tag{A.6}$$

We shall need the Poisson summation formula on $L^2(\mathbb{R})$ stated as a relation between Fourier transformation and the sampling- and periodization operators. A proof can be found in [12, p. 105].

Theorem 28. *The Poisson summation formula:*

$$\mathcal{P}_M = \mathcal{F}^{-1}_{[0,M]} \mathcal{S}_{1/M} \mathcal{F}_\mathbb{R} \textit{ on } S_0(\mathbb{R}) \tag{A.7}$$

The following simple lemma [3, Lemma 5.3.3] shall be used frequently.

Lemma 29. *Let T be a unitary operator on $\mathcal{H}$, and assume that $\{e_j\}$ is a frame for $\mathcal{H}$. Then $\{Te_j\}$ is also a frame for $\mathcal{H}$ with the same frame bounds.*

The dual frames that have Gabor structure are characterized by the Wexler-Raz relations:

Theorem 30 (Wexler-Raz)**.** *If g and γ both generates Gabor systems as in Definition 9 with finite upper frame bounds, then they are dual windows if and only if*

$L^2(\mathbb{R})$	$\frac{1}{\alpha\beta}\left\langle\gamma, M_{m/\alpha}T_{n/\beta}g\right\rangle = \delta_m\delta_n, \quad m,n\in\mathbb{Z}$
$l^2(\mathbb{Z})$	$\frac{M}{a}\left\langle\gamma, M_{m/a}T_{nM}g\right\rangle = \delta_m\delta_n, \quad m=0,...,a-1, n\in\mathbb{Z}$
$L^2([0,L])$	$\frac{N}{b}\left\langle\gamma, M_{mN}T_{nM/L}g\right\rangle = \delta_m\delta_n, \quad m\in\mathbb{Z}, n=0,...,b-1$
$\mathbb{C}^L$	$\frac{MN}{L}\left\langle\gamma, M_{mN}T_{nM}g\right\rangle = \delta_m\delta_n, \quad m=0,...,a-1, n=0,...,b-1$

Proof of the original result for $L^2(\mathbb{R})$ and $\mathbb{C}^L$ can be found in [23]. More rigorous proofs with a minimal sufficient condition appear in [15, 5] and equally in [14]. Also see [18, Subsecs. 1.4.2 and 1.6.4].

Combining Lemma 29 with the relations (A.1), (A.2) and (A.3)-(A.6) yields the following well known results. The first relation appears as [12, 6.36].

Lemma 31. *Dual Gabor frames under Fourier transforms and dilations.*

$\mathcal{F}_\mathbb{R}$: Let γ^0 be the canonical dual window of (g,α,β), $g\in L^2(\mathbb{R})$. The canonical dual window of $(\mathcal{F}_\mathbb{R}g,\beta,\alpha)$ is $\mathcal{F}_\mathbb{R}\gamma^0$.

$\mathcal{F}_\mathbb{Z}$: Let γ^0 be the canonical dual window of $\left(g,a,\frac{1}{M}\right)$, $g\in l^2(\mathbb{Z})$. The canonical dual window of $\left(\mathcal{F}_\mathbb{Z}g,\frac{1}{M},a\right)$ is $\mathcal{F}_\mathbb{Z}\gamma^0$.

$\mathcal{F}_{[0,L]}$: Let γ^0 be the canonical dual window of (g,a,b), $g\in L^2([0,L])$. The canonical dual window of $\left(\mathcal{F}_{[0,L]}g,b,\frac{a}{L}\right)$ is $\mathcal{F}_{[0,L]}\gamma^0$.

$\mathcal{F}_L$: Let γ^0 be the canonical dual window of (g,a,b), $g\in\mathbb{C}^L$. The canonical dual window of $(\mathcal{F}_Lg,b,a)$ is $\mathcal{F}_L\gamma^0$.

D_d on $L^2(\mathbb{R})$: Let γ^0 be the canonical dual window of (g,α,β), $g\in L^2(\mathbb{R})$. The canonical dual window of $\left(D_dg,\frac{\alpha}{d},\beta d\right)$ is $D_d\gamma^0$.

D_d on $L^2([0,L])$: Let γ^0 be the canonical dual window of (g,a,b), $g\in L^2([0,L])$. The canonical window of $\left(D_dg,\frac{a}{d},b\right)$ is $D_d\gamma^0$.

The proofs can be found by direct calculation. They are very simple, and almost identical. The only difference is which of the commutation relations (A.1), (A.2) and (A.3)-(A.6) to use.

Proof of Proposition 12. Define $c\in l^2(\mathbb{Z}\times\mathbb{Z})$ and $d\in l^2(\{0,...,a-1\}\times\mathbb{Z})$ by

$$\begin{aligned}
c(m,n) &= \left\langle\gamma, M_{m/\alpha}T_{n/\beta}g\right\rangle_{L^2(\mathbb{R})}, \quad m,n\in\mathbb{Z},\\
d(m,n) &= \left\langle\mathcal{S}_{\alpha/a}\gamma, M_{m/a}T_{nM}\mathcal{S}_{\alpha/a}g\right\rangle_{l^2(\mathbb{Z})}\\
&= \left\langle\mathcal{S}_{1/M\beta}\gamma, M_{m/a}T_{nM}\mathcal{S}_{1/M\beta}g\right\rangle_{l^2(\mathbb{Z})}, \quad m=0,...,M-1, n\in\mathbb{Z}
\end{aligned}$$

It is well known, that if $g,\gamma\in S_0(\mathbb{R})$ then (g,α,β) and (γ,α,β) have finite upper frame bounds, see e.g. [12, Chapter 6]. Since g,γ are dual windows, they satisfy the Wexler-Raz condition for $L^2(\mathbb{R})$, Theorem 30:

$$\frac{1}{\alpha\beta}c(m,n) = \delta_m\delta_n, \quad m,n\in\mathbb{Z}.$$

We wish to show that $\mathcal{S}_{\alpha/a}g$ and $\mathcal{S}_{\alpha/a}\gamma$ satisfies the Wexler-Raz condition for $l^2(\mathbb{Z})$:

$$\begin{aligned} & \frac{M}{a}\left\langle \mathcal{S}_{1/M\beta}\gamma, M_{m/a}T_{nM}\mathcal{S}_{1/M\beta}g\right\rangle_{l^2(\mathbb{Z})} \\ = & \frac{M}{a}d(m,n) \\ = & \delta_m\delta_n, \quad m = 0,...,a-1, n\in\mathbb{Z} \end{aligned}$$

By Theorem 10, $\left(\mathcal{S}_{\alpha/a}g, a, \frac{1}{M}\right)$ and $\left(\mathcal{S}_{\alpha/a}\gamma, a, \frac{1}{M}\right)$ also has finite upper frame bounds. By Proposition 11:

$$\begin{aligned} \frac{M}{a}d(m,n) &= \sum_{j\in\mathbb{Z}} \frac{1}{\alpha\beta}c(m-ja,n) \\ &= \sum_{j\in\mathbb{Z}} \delta_{m-ja}\delta_n \\ &= \delta_m\delta_n, \quad \forall m = 0,...,a-1, n\in\mathbb{Z}. \end{aligned}$$

This shows that $\mathcal{S}_{\alpha/a}g$ and $\mathcal{S}_{\alpha/a}\gamma$ satisfies the Wexler-Raz condition for $l^2(\mathbb{Z})$, and therefore they are dual windows. □

A.2. **Proofs of section 3.2.**

Proof of Theorem 13. By Lemma 29 and Lemma 31, $(\mathcal{F}g, \beta, \alpha)$ is also a Gabor frame for $L^2(\mathbb{R})$ with frame bounds A and B and canonical dual window $\mathcal{F}\gamma^0$.

Since $\hat{g} \in S_0(\mathbb{R})$, Theorem 10 can be used: $\left(\mathcal{S}_{\beta/b}\mathcal{F}g, b, \frac{1}{N}\right)$ is a Gabor frame for $l^2(\mathbb{Z})$ with frame bounds A and B and canonical dual window $\mathcal{S}_{\beta/b}\mathcal{F}\gamma^0$.

By Lemma 29 and Lemma 31, $\left(\mathcal{F}^{-1}_{[0,\frac{b}{\beta}]}\mathcal{S}_{\beta/b}\mathcal{F}g, a, b\right)$ is a Gabor frame for $L^2([0,\frac{b}{\beta}])$ with frame bounds A and B and canonical dual window $\mathcal{F}^{-1}_{[0,\frac{b}{\beta}]}\mathcal{S}_{\beta/b}\mathcal{F}\gamma^0$.

By the Poisson summation formula (A.7) then $\mathcal{F}^{-1}_{[0,\frac{b}{\beta}]}\mathcal{S}_{\beta/b}\mathcal{F} = \mathcal{P}_{b/\beta}$. From this the result follows. □

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