Nonstationary Gabor Frames

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Abstract:

To overcome the limitation induced by the fixed time-frequency resolution over the whole time-frequency plane of Gabor frames, we propose a simple extension of the Gabor theory leading to the construction of frames with time-frequency resolution evolving over time or frequency. We describe the construction of such frames and give the explicit formulation of the canonical dual frame for some conditions. We illustrate the interest of the method on a simple example.

1. Introduction

Gabor analysis [7] is widely used for applications in signal processing. For some of these applications, which include processing of signals using Gabor frame multipliers [6, 1], the rigid construction of the Gabor atoms results in important limitations on the signal analysis and processing ability of the associated schemes. The Gabor transform uses time-frequency atoms built by translation over time and frequency of a unique prototype function, leading to a signal decomposition having a fixed time-frequency resolution over the whole time-frequency plane. This can be very restricting when dealing with signals with characteristics changing over the time-frequency plane. For example, this led some people to prefer the use of alternative decompositions with time-frequency resolution evolving with frequency in some applications, to better fit the feature of interest of the signal. Examples of such decompositions are the wavelet transform [5] or the decompositions using filter banks based on perceptive frequency scales for processing of audio signals, as for example gammatone filters [9].

A case for which the limitation induced by the constant time-frequency resolution of the Gabor transform can be seen is shown on the didactic example of Figure 1. On this figure, two spectrograms of the same glockenspiel signal are represented. These spectrograms are obtained by plotting the square absolute value of the Gabor coefficients using a color scale with a level coding in dB. Both spectrograms are obtained from the Gabor coefficients using the same type of window, but using two different window lengths. We see that the signal contains two very contrasting types of components:

- at the beginning of the notes, the signal presents sharp attacks which are spread in frequency, but very localized in time,
- during the resonance of the notes, the signal contains quasi-sinusoidal components which are spread in time, but very localized in frequency.

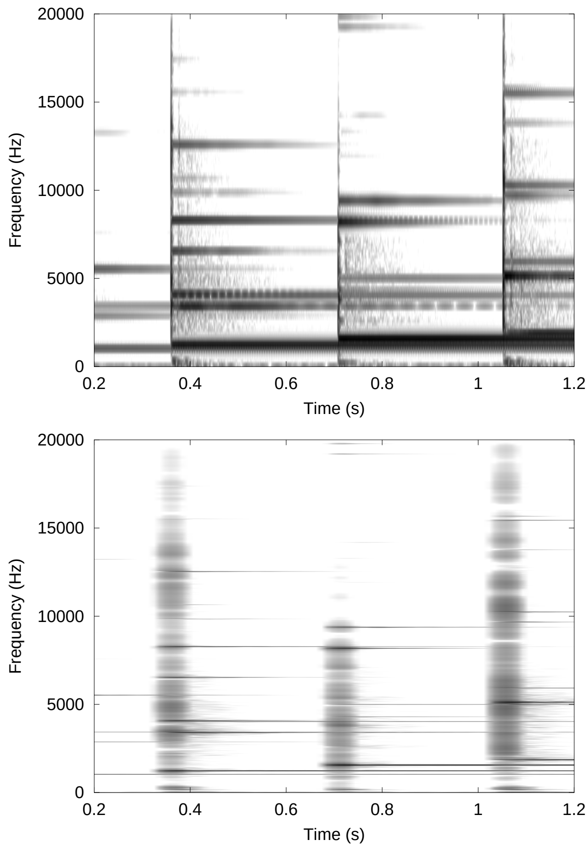

Figure 1: Two spectrograms of the same glockenspiel signal obtained using two different window lengths. On the top plot, a narrow window of 6 ms is used, on the bottom plot, a wide window of 93 ms is used.

We see that the use of the narrow window is well suited for the analysis and processing of the attacks, leading to a very sparse decomposition for these components, but gives an unsatisfying representation of the resonance, as the different sinusoidal components are not resolved. On the other hand, the wide window gives a good representation of the resonance part, but a blurred representation of the attacks. For this example, it appears that if we want to build an optimised scheme for processing of both attacks and the resonances at the same time, it would be suitable to be able to adapt the time-frequency resolution locally for the different types of components.

The purpose of this paper is to describe one way to achieve this goal. For this, we show that, while staying in the context of frame theory [2, 4], the standard Gabor theory can be easily extended to provide some freedom of evolution of the time-frequency resolution of the decomposition in either time or frequency. Furthermore, this extension is well suited for applications as it can easily be implemented using fast algorithm based on fast Fourier transform [12].

We first describe the construction of the frames in Section 2., and then illustrate in Section 3. the potential of the approach on the preceding example of Figure 1.

2. Construction of the frames

2.1 Resolution evolving over time

As opposed to standard Gabor analysis, we replace time translation for the construction of atoms by the use of different windows for the different sampling positions in time. For each time position we still build atoms by regular frequency modulation. So using a set of functions $\{g_n\}_{n\in\mathbb{Z}}$ of $\mathbf{L}^2(\mathbb{R})$, for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we define atoms of the form:

$$g_{m,n}(t) = g_n(t) e^{i2\pi m b_n t}.$$

In practice we will choose each window g_n centered around a time a_n, and it will typically be constructed by translating a well localized window centered around 0 by a_n, as in the standard Gabor scheme, but with the possibility to vary the window g_n for each position a_n. Thus the sampling of the time-frequency plane is done on a grid which is irregular over time, but regular over frequency. Figure 2 shows an example of such a sampling grid. It can be noted that some results exist in Gabor theory for semi-regular sampling grids, as for example in [3]. Our study here uses a more general setting, as the sampling grid is in general not separable, and more importantly, the window can evolve over time.

In this case, the coefficients of the decomposition are given by:

$$c_{m,n} = \langle f, g_{m,n} \rangle,$$

and the frame operator is given by:

$$\mathbf{S}f = \sum_m \sum_n \langle f, g_{m,n} \rangle g_{m,n}.$$

The frame operator can be described by its kernel K given the following relation, which holds at least in a weak sense:

$$\mathbf{S}f(s) = \int K(t,s) f(t) \mathrm{d}t.$$

Here the kernel K simplifies according to the following

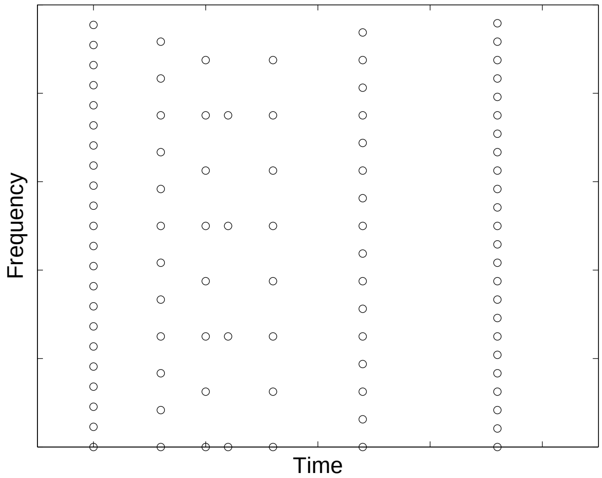

Figure 2: Example of sampling grid of the time-frequency plane when building a decomposition with time-frequency resolution evolving over time.

relations:

$$\begin{aligned} K(t,s) &= \sum_m \sum_n \overline{g_n(t)} g_n(s) e^{i2\pi m b_n (s-t)} \\ &= \sum_n \overline{g_n(t)} g_n(s) \sum_m e^{i2\pi m b_n (s-t)} \\ &= \sum_n \frac{1}{b_n} \overline{g_n(t)} g_n(s) \sum_k \delta\left(s - t - \frac{k}{b_n}\right) \end{aligned}$$

thus,

$$\mathbf{S}f(s) = \sum_k \sum_n \frac{1}{b_n} \overline{g_n\left(s - \frac{k}{b_n}\right)} g_n(s) f\left(s - \frac{k}{b_n}\right)$$

In general, the inversion of $\mathbf{S}$ is not obvious. However we can identify a special case, which is analog to the "painless" case in standard Gabor analysis [8], for which the expression of $\mathbf{S}$ simplifies.

More precisely, we suppose from now on that for every $n \in \mathbb{Z}$, the function g_n has a limited time support supp $g_n = [c_n, d_n]$ such that $d_n - c_n < \frac{1}{b_n}$. Due to this support condition, the terms of the summation over k in the preceding equation are 0 for $k \neq 0$ and the frame operator $\mathbf{S}$ becomes a multiplication operator:

$$\mathbf{S}f(s) = \sum_n \frac{1}{b_n} |g_n(s)|^2 f(s).$$

In this case the invertibility of the frame operator is easy to check and the system of functions $g_{m,n}$ forms a frame for $\mathbf{L}^2(\mathbb{R})$ if and only if $\forall t \in \mathbb{R}$, $\sum_n \frac{1}{b_n} |g_n(t)|^2 \simeq 1$. When this condition is fulfilled, the canonical dual frame elements are given by:

$$\tilde{g}_{m,n}(t) = \frac{g_n(t)}{\sum_k \frac{1}{b_k} |g_k(t)|^2} e^{i2\pi m b_n t},$$

and the associated canonical tight frame elements can be calculated by:

$$\dot{g}_{m,n}(t) = \frac{g_n(t)}{\sqrt{\sum_k \frac{1}{b_k} |g_k(t)|^2}} e^{i2\pi m b_n t}.$$

2.2 Resolution evolving over frequency

An analog construction is possible with a sampling of the time-frequency plane irregular over frequency, but regular over time. An example of the sampling grid in such a case is given on Figure 3.

In this case, we introduce a family of functions $\{h_m\}_{m\in\mathbb{Z}}$ of $\mathbf{L}^2(\mathbb{R})$, and for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$, we define atoms of the form:

$$h_{m,n}(t) = h_m(t - na_m).$$

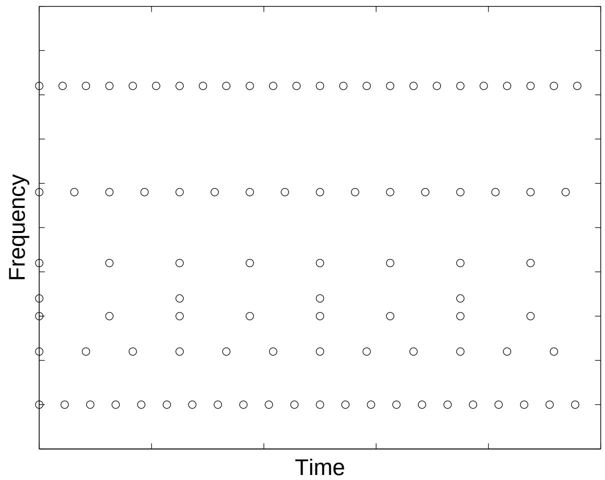

Figure 3: Example of sampling grid of the time-frequency plane when building a decomposition with time-frequency resolution evolving over frequency.

In practice we will choose each function h_m as a well localized pass-band function having a Fourier transform centered around some frequency b_n.

In this case the frame operator is given by:

$$\mathbf{T}f = \sum_m \sum_n \langle f, h_{m,n}\rangle h_{m,n},$$

and the problem is completely analog to the preceding up to a Fourier transform, as we have:

$$\widehat{\mathbf{T}f} = \sum_m \sum_n \langle \widehat{f}, \widehat{h_{m,n}}\rangle \widehat{h_{m,n}},$$

and $\widehat{h_{m,n}} = \widehat{h_m}(\nu)e^{-i2\pi na_m\nu}$. So the preceding computation can be done, working on the Fourier transforms of the involved functions instead of directly on the functions.

Now the "painless" case appears when we suppose that for every $m \in \mathbb{Z}$, the function $\widehat{h_n}$ has a limited frequency support supp $\widehat{h_n} = [e_n, f_n]$ such that $f_n - e_n < \frac{1}{a_n}$. Then the following expression holds:

$$\widehat{\mathbf{T}f}(\nu) = \sum_m \frac{1}{a_m}|\widehat{h_m}(\nu)|^2 \widehat{f}(\nu),$$

and the system of functions $h_{m,n}$ forms a frame of $\mathbf{L}^2(\mathbb{R})$ if and only if $\forall \nu \in \mathbb{R}$, $\sum_n \frac{1}{a_m}|\widehat{h_m}(\nu)|^2 \simeq 1$.

The associated canonical dual and tight frame can be computed as preceding, with the addition of an inverse Fourier transform.

2.3 Implementation

For the practical implementation, we have developed the equivalent theory in a finite discrete setting, that is to say working with complex vectors as signals. This theory won't be described here due to lack of space, but the construction is very similar to the one described in 2.1 and 2.2 and leads to a frame matrix which simplifies to a diagonal matrix in the "painless" case, suitable for applications.

The implementation is then very similar to the implementation of the standard Gabor case and can exploit fast Fourier transform algorithms for efficiency. The only differences compared to standard Gabor implementation are due to the fact that the storage of coefficients requires more advanced storage structures due to the irregularity of the time-frequency sampling grid, and that the computation of the dual window must be performed for every time position resulting in a slight increase in computational cost.

3. Example

The possibility to build a decomposition with time-frequency resolution evolving over time can be exploited to solve the problem described in example of Section 1. illustrated by Figure 1. For the corresponding glockenspiel signal, as we have seen before, the use of narrow window is suitable for the attacks of the notes, while a wide window should be used for the resonances. Figure 4 shows a representation built with our approach using a narrow window of 6 ms for the attacks and a wide window of 93 ms for the resonance. The frame used for this figure is a tight frame. It should be noticed that the evolution of the window size between the two target window lengths is smoothed in order to ensure that the atoms used for the decomposition maintain a "nice" shape, in the sense of having a good time-frequency concentration. This ensures the easy interpretability of the decomposition, especially for processing using frame multipliers.

This figure gives an idea of the type of decompositions that can be constructed with our approach and should be compared to the decomposition obtained using standard Gabor analysis on Figure 1. With our approach, it becomes possible to have a simultaneous good representation of both types of components of this signal while keeping the same processing ability than with standard Gabor.

We see that our approach allows to build decompositions with better time-frequency localization of the signal energy. This can be helpful for many processing tasks, in particular to reduce artifacts in component extraction or denoising.

4. Conclusion

Our approach enables the construction of frames with flexible evolution of time-frequency resolution over time or frequency. The resulting frames are well suited for applications as they can be implemented using fast algorithms, at a computational cost close to standard Gabor frames.

Exploiting evolution of resolution over time, the proposed approach can be of particular interest for applica-

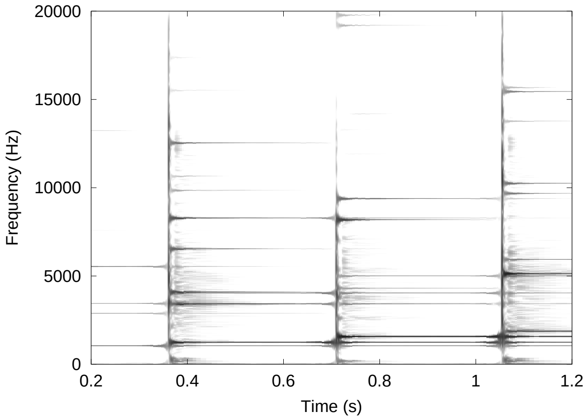

Figure 4: Spectrogram of the same glockenspiel signal as in Figure 1 using a nonstationary Gabor decomposition.

tions where the frequency characteristics of the signal are known to evolve significantly with time. Order analysis [11], in which the signal analyzed is produced by a rotating machine having evolving rotating speed, is an example of such application.

Exploiting evolution of resolution over frequency, the presented approach could be valuable for applications requiring the use of a tailored non uniform filter bank. In particular, it can be used to build filter banks following some perceptive frequency scale.

One difficulty when using our approach is to adapt the time-frequency resolution to the evolution of the signal characteristics. If prior knowledge is available, this can be done by hand, as for the example of Figure 4. But to go further, our approach could be extended to construct an adaptive decomposition of the signal by automatically adapting the resolution to the signal. To achieve this, we plan to investigate the possibility to couple our approach with the use of sparsity criterion as proposed in [10]. The general idea would then be to consider time segments of the signal, and for each time segment compare the sparsity criterion obtained for Gabor transforms computed with different possible windows. We would then use in our decomposition the window corresponding to the best criterion for each time segment, leading to a decomposition optimizing the sparsity of the decomposition over time.

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