THE POLE BEHAVIOUR OF THE PHASE DERIVATIVE OF THE SHORT-TIME FOURIER TRANSFORM

PETER BALAZS[A)], DOMINIK BAYER[A)], FLORENT JAILLET[B)] AND PETER SØNDERGAARD[C)]

Abstract. The short-time Fourier transform (STFT) is a time-frequency representation widely used in audio signal processing. Recently it has been shown that not only the amplitude, but also the phase of this representation can be successfully exploited for improved analysis and processing. In this paper we describe a rather peculiar pole phenomenon in the phase derivative, a recurring pattern that appears in a characteristic way in the neighborhood around any of the zeros of the STFT, a negative peak followed by a positive one. We describe this phenomenon numerically and provide a complete analytical explanation.

1. Introduction

The *short-time Fourier transform* (STFT) [5, 11] is a time-frequency representation widely used in audio signal processing. A common definition of the STFT[1] is

$$V(f,g)(x,\omega) = \int f(t)\overline{g(t-x)}e^{-2\pi i\omega t}\,dt. \tag{1}$$

The STFT $V(f,g)(x,\omega)$ provides information about the frequency content of the signal f at time x and frequency ω. The analyzing window g determines the resolution in time and frequency.

The interpretation of the modulus of the STFT is relatively easy, considering the fact that the spectrogram (defined as the square absolute value of the STFT) can be interpreted as a time-frequency distribution of the signal energy. This interpretation led to the important success of the STFT in signal processing. In particular, it has been widely used

Date: July 17, 2014.

[A)] Acoustics Research Institute, Austrian Academy of Sciences, Vienna A-1040, Austria;
[B)] Institut de Neurosciences de la Timone UMR 7289, Aix Marseille Université, CNRS, 13385 cedex 5, Marseille, France;
[C)] Oticon A/S, 2765 Smørum, Denmark;

[1]This is the *frequency-invariant* STFT.

for applications in speech processing and acoustics as a graphical tool for signal analysis [17].

But the interpretation of the phase of the STFT is less obvious, and was thus hardly considered in applications for some time. The phase can be of particular interest for certain applications, as illustrated by important applications such as phase vocoder [10, 7] or reassignment [14, 1]. In digital image processing it is well known that the phase information of the discrete Fourier transform is at least as important as the amplitude information. In [15] it is shown that as long as the phase of the discrete Fourier transform of an image is retained and the amplitude is set to 1, the image can still be recognized. Similar effects can also be shown for acoustic signal depending on the parameters of the STFT [4].

For applications modifying the STFT coefficients, phase information is essential again. For this type of applications, in particular for applications using Gabor frame multipliers [9, 3] which motivated the present study, better understanding of the structure of the phase is necessary to improve the processing possibilities.

The phase of the STFT is usually not considered directly. In fact, it is more interesting to consider the phase derivative over time or frequency. Indeed, these quantities appear naturally in the context of reassignment [1] and manipulations of phase derivative over time is the idea behind the phase vocoder [7]. Their interpretation is easier, as the derivative of phase over time can be interpreted as local instantaneous frequency while the derivative of the phase over frequency can be interpreted as a local group delay.

To numerically compute the local instantaneous frequency, an unwrapping of the phase is needed to avoid discontinuities. This is the classical method used in [7, 14]. Another method was found in [1]:

$$\frac{\partial}{\partial x} \arg(V(f,g)(x,\omega)) = \operatorname{Im}\left(\frac{V(f,g')(x,\omega)\overline{V(f,g)(x,\omega)}}{|V(f,g)(x,\omega)|^2}\right), \tag{2}$$

with $g'(t) = \frac{dg}{dt}(t)$. The benefit of this method is that is does not require unwrapping, instead the phase derivative is computed by pointwise operations using a second STFT based on the derivative of the window.

To understand the phase of the STFT more thoroughly, in particular for applications dealing with multipliers, see e.g. [18, 19, 16], we conducted related extensive numerical experiments. In the process we observed a rather peculiar phenomenon in the phase derivative, a recurring pattern that appears in a similar way in the neighborhood around any of the zeros of the STFT. The behavior of the phase derivative close to the singularity always shows the same characteristic shape, i.e. a negative peak followed by a positive one. We describe this phenomenon and provide a complete analytical explanation.

This paper is organized as follows: In Section 2 we report the numerical results. In Section 3 we give a short, instructive, analytical examples for this behaviour. In Section 4 we give the full analytical results.

Results in this paper have partly been reported at a conference [13], and a preprint of it has already been cited in [2].

2. Numerical Observations

For noise, naturally only statistical properties of the phase are accessible. Some interesting results for the phase derivative have been shown in the context of reassignment. In particular the study of the distribution of the phase derivative values appear in [8]. There, the following result is given. We consider a zero-mean Gaussian analytic white noise f such that

$$\mathrm{E}[\mathrm{Re}(f(t)) \cdot \mathrm{Re}(f(s))] = \mathrm{E}[\mathrm{Im}(f(t)) \cdot \mathrm{Im}(f(s))] = \frac{\sigma^2}{2}\delta(t-s) \tag{3}$$

and $\mathrm{E}[f(t)f(s)] = 0$ for any $(t,s) \in \mathbb{R}^2$, with its real and imaginary parts a Hilbert transform pair. Using a Gaussian window given by $g(t) = e^{-\pi\frac{t^2}{2\sigma^2}}$, the phase derivative over time of $V(f,g)$ is a random variable with distribution of the form:

$$\rho(v) = \frac{1}{2(1+v^2)^{\frac{3}{2}}}. \tag{4}$$

This distribution is shown in Figure 1. As can be seen, it is a quite "peaky" distribution, indicating that the values of the phase derivative are mainly values close to zero, with some rare values with higher absolute values.

But the information about the distribution of the values of the phase derivative does not give any clue about the spatial distribution of these values in the time-frequency plane. As accessing this information theoretically seems particularly difficult, we conducted systematic numerical experiments to study this spatial distribution.

For this, we need to compute the derivative of the phase in discrete settings. We used the expression (2) to compute the phase derivative.

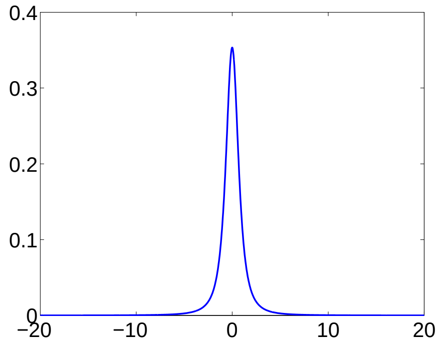

FIGURE 1. Distribution of the values of the phase derivative over time of the STFT for a white Gaussian noise.

We see on this formula that we will face numerical difficulties when the denominator $V(f,g)(x,\omega)$ is close to zero. But using double precision, these problems appear for really small values of the modulus (on the order of 10^{-13}), which allows us to reliably observe the values of the phase derivative even close to the zeros of the STFT. In the figures of this paper, the phase derivative values are ignored and represented as white at the points where the value of the modulus is too small.

In the following, we are going to present numerical experiments about the behavior of the phase derivative around the zeros of the STFT. Note that those experiments have already been presented at a conference [13].

The results of our experiments are illustrated by Figure 2. As can be seen on this figure, the time-frequency distribution of the values appears to be highly structure, as e.g. noted in [12]. The values of the phase derivative with high absolute values are concentrated around several time-frequency points, which can be identified as the zeros of the transform when looking at the modulus. Furthermore, the shape of the phase derivative seems to be very similar in the neighbourhood of the zeros, with a typical pattern repeating at each zero. This typical pattern is represented on the third image of Figure 2. When going from low to high frequencies, it presents a negative peak followed by a positive one.

This phenomenon is related to the fact that the STFT of white noise is a correlated process, with a correlation determined by the window through the reproducing kernel of the transform (see part 6.2.1 of [5]). It is thus interesting to study the influence of the window choice on the observed structure of the phase derivative.

This influence is illustrated on Figure 3. Comparing Figure 2 and the first display of Figure 3, we can observe the effect of scaling of

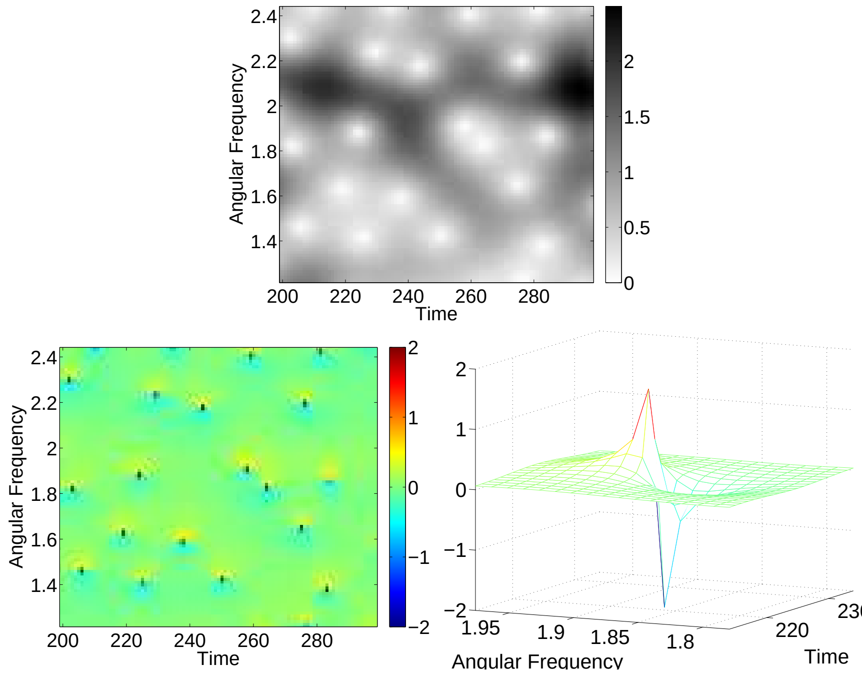

FIGURE 2. Observation for a Gaussian white noise, using a Gaussian window. Top: modulus of the STFT. Bottom-left: derivative over time of the phase of the STFT using the definition (1). Bottom-right: mesh plot of the derivative over time of the phase in the neighbourhood of a zero of the STFT.

the window (or change of window length). We see that narrowing the window results in similar patterns around the zeros, but with a scaled shape: the resulting pattern is narrower over time, but wider over frequency.

Figure 3 also shows the influence of the window type. We see that for windows with a worse time-frequency concentration than the Gaussian window, the structure is more complicated. On the representation using a Hamming window, we still observe repeating patterns at the zeros of the transform, but the variability of the shape of this pattern seems higher, and the pattern orientation slightly varies, whereas it is fixed in the case of a Gaussian window.

For the case of the rectangular window, the zeros of the STFT form a more complicated, extended structures. This leads to much more variable patterns. Yet, we still, interestingly, observe that the values of the phase derivative with high absolute values concentrate around the zeros of the transform, whereas the phase derivative is close to zero in the regions of the STFT where the modulus is high.

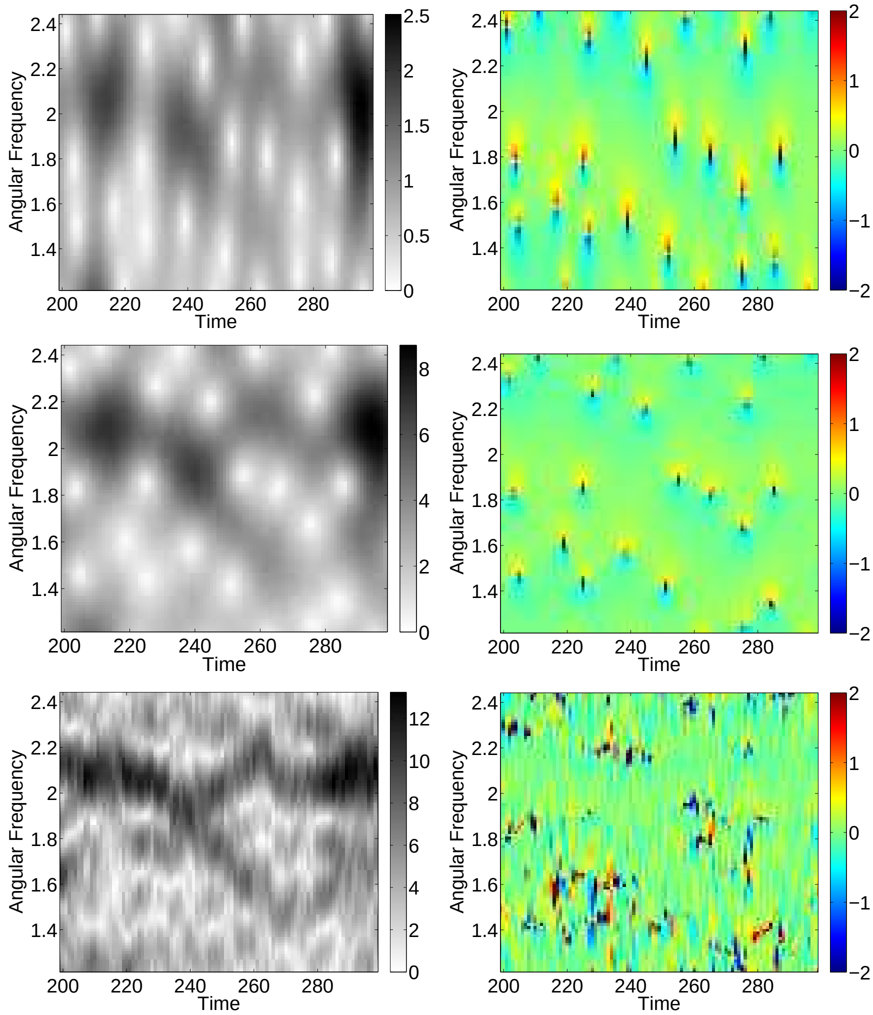

FIGURE 3. Influence of the window when analyzing a frozen Gaussian white noise. For three different windows, on the left, modulus of the STFT, on the right, derivative over time of the phase of the STFT using the definition (1). From top to bottom, the windows are: a narrower Gaussian window, a Hamming window, a rectangular window.

The behaviour that we observe for the noise is not specific to this kind of signal. Indeed further experiments on other synthesized and recorded complex sounds showed that the same characteristics can be observed for all signals: the values of the phase derivative of high absolute value are concentrated in the neighbourhood of the zeros of the STFT, and for "nice" windows, a specific pattern appears in this neighbourhood.

3. A Simple Explicit Analytic Example

In this section we give a simple analytical example for which we can explicitly compute the phase derivative.

Considering the signal given by

$$f(t) = e^{2\pi i\omega_1 t} + e^{2\pi i\omega_2 t} \tag{5}$$

and using a Gaussian window $g(t) = e^{-\pi\frac{t^2}{2\sigma^2}}$, we can explicitly compute the expression of the STFT, which results in the formula:

$$V(f,g)(x,\omega) = e^{-2\pi ix(\omega-\omega_1)}e^{-2\pi\sigma^2(\omega-\omega_1)^2} + e^{-2\pi ix(\omega-\omega_2)}e^{-2\pi\sigma^2(\omega-\omega_2)^2}. \tag{6}$$

The zeros of this STFT are the points of coordinates (x_k, ω_{mid}) in the time-frequency plane, with $\omega_{mid} = \frac{\omega_1+\omega_2}{2}$ and $x_k = \frac{1+2k}{2(\omega_1-\omega_2)}$ for $k \in \mathbb{Z}$.

The expression of the phase derivative for this signal, given in part VI-12 of [6], is:

$$\frac{\partial}{\partial x}\arg(V(f,g)(x,\omega)) = 2\pi\left(\omega_{mid} - \omega + \delta\tanh(s)\frac{1+\tan^2(2\pi\delta x)}{1+\tan^2(2\pi\delta x)\tanh^2(s)}\right) \tag{7}$$

with $\delta = \frac{\omega_2-\omega_1}{2}$ and $s = 4\pi\sigma^2(\omega-\omega_{mid})\delta$.

The plot of this function around one of the zeros is visible in Figure 4. We see the pattern that was already observed in the previous section.

For an extended treatment of this analytic example, see [2], where the authors are already referring to a preprint of our present work.

4. Analytical Results

In this section, we denote by M_h the modulation operator $M_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $f(t) \mapsto M_h f(t) := e^{-2\pi iht}f(t)$, and by T_h the translation operator $T_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $f(t) \mapsto T_h f(t) := f(t-h)$ (with $h \in \mathbb{R}$). The set $\mathcal{S}(\mathbb{R})$ is the Schwartz class of rapidly decaying functions. The Fourier transform is denoted by $\mathcal{F}$.

4.1. Regularity Properties of the STFT.

Definition 4.1. *Define the (unbounded) operator P on $L^2(\mathbb{R})$ as the multiplication operator*

$$Pf(t) := 2\pi it \cdot f(t)$$

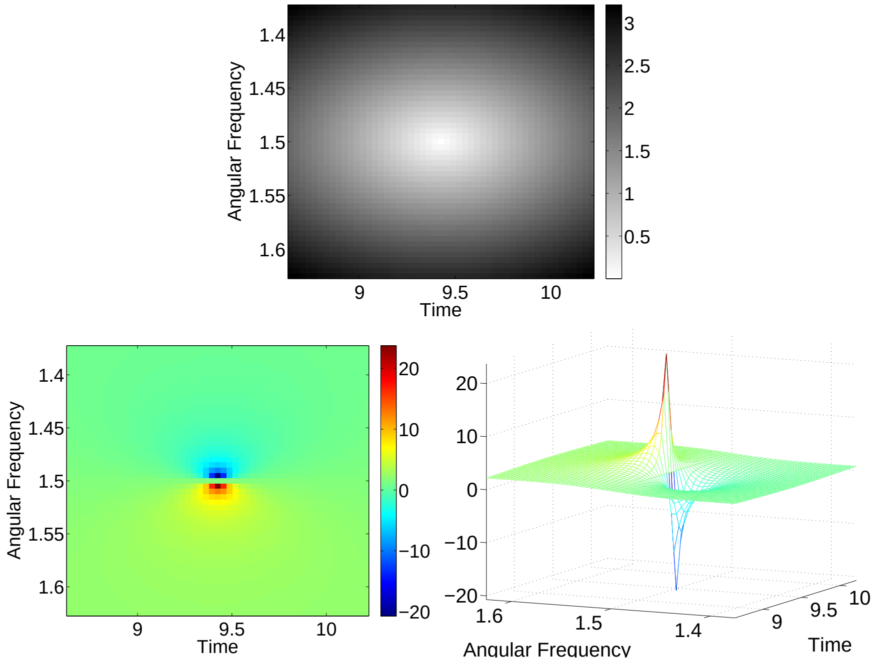

FIGURE 4. Observation for the signal defined in (5). Top: modulus of the STFT. Bottom: derivative over time of the phase of the STFT according to (7) represented as an image (left) and as a mesh (right).

with domain

$$Dom(P) := \{f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |t\, f(t)|^2\, dt < \infty\} \subset L^2(\mathbb{R}).$$

Further, define the (unbounded) operator

$$Q := \mathcal{F}^{-1} P \mathcal{F}$$

(where $\mathcal{F}$ denotes the Fourier transform) with domain

$$Dom(Q) := \{f \in L^2(\mathbb{R}) : \mathcal{F} f \in Dom(P)\} \subset L^2(\mathbb{R}).$$

In quantum mechanics, these operators are essentially the momentum and position operator, respectively. The operator P (and thus also Q) are clearly densely-defined, since $\mathcal{S}(\mathbb{R}) \subset \mathrm{Dom}(P)$ (and $\mathcal{F}^{-1}\mathcal{S}(\mathbb{R}) = \mathcal{S}(\mathbb{R}) \subset \mathrm{Dom}(Q)$). It can be shown that P and Q are closed unbounded operators and that iP and iQ are self-adjoint.

We collect basic properties of these operators in the following lemma.

Lemma 4.2. *The operators P and Q have the following properties:*

(i) $\mathcal{F}Q = P\mathcal{F}$ *on* $Dom(Q)$, $Q\mathcal{F} = -\mathcal{F}P$ *on* $Dom(P)$;

(ii) *Q is a (maximal extension of a) differential operator, more precisely: if $f \in \mathcal{S}(\mathbb{R})$, then $Qf(t) = \frac{d}{dt}f(t)$.*

The next lemma is in essence a well-known result from the theory of operator (semi-)groups; it gives the infinitesimal generators of the modulation and translation group, respectively. It can be proved in a straight-forward way:

Lemma 4.3. *Let $f \in Dom(P)$, then*

$$\|\frac{1}{h}(M_h - Id)f - Pf\|_{L^2} \to 0$$

for $h \to 0$.
Let $f \in Dom(Q)$, then

$$\|\frac{1}{h}(T_h - Id)f + Qf\|_{L^2} \to 0$$

for $h \to 0$.

We can now prove a regularity result for the short-time Fourier transform.

Proposition 4.4. *Let $f, g \in L^2(\mathbb{R})$.*

(i) *If f belongs to $Dom(P)$, then $V(f,g)$ has a continuous partial derivative with respect to the second argument ω, and we have*

$$\frac{\partial}{\partial \omega} V(f,g)(x,\omega) = -V(Pf,g)(x,\omega).$$

(ii) *If g belongs to $Dom(Q)$, then $V(f,g)$ has a continuous partial derivative with respect to the first argument x, and we have*

$$\frac{\partial}{\partial x} V(f,g)(x,\omega) = -V(f,Qg)(x,\omega).$$

Proof. Assume $f \in \mathrm{Dom}(P)$. Then, by the preceding lemma,

$$\begin{aligned}\frac{1}{h}(V(f,g)(x,\omega+h) - V(f,g)(x,\omega)) &= \langle f, \frac{M_h - Id}{h} M_\omega T_x g\rangle \\ &= \langle \frac{M_{-h} - Id}{h} f, M_\omega T_x g\rangle \\ &\overset{h\to 0}{\longrightarrow} \langle -Pf, M_\omega T_x g\rangle = -V(Pf,g)(x,\omega),\end{aligned}$$

which is a continuous function on $\mathbb{R}^2$.
If $g \in \mathrm{Dom}(Q)$, then, analogously,

$$\begin{aligned}\frac{1}{h}(V(f,g)(x+h,\omega) - V(f,g)(x,\omega)) &= \langle f, M_\omega T_x \frac{T_h - Id}{h} g\rangle \\ &\overset{h\to 0}{\longrightarrow} \langle f, -M_\omega T_x Q g\rangle = -V(f,Qg)(x,\omega).\end{aligned}$$

□

Using $V(f,g)(x,\omega) = e^{-2\pi i x\omega}\overline{V(g,f)(-x,-\omega)}$, the partial derivatives of $V(f,g)$ exist if and only if those of $V(g,f)$ exist. We may thus change the roles of f and g and have the

Corollary 4.5. *Let $f, g \in L^2(\mathbb{R})$.*

(i) *If g belongs to Dom(P), then $V(f,g)$ has a continuous partial derivative with respect to the second argument ω, and we have*

$$\frac{\partial}{\partial\omega}V(f,g)(x,\omega) = V(f,Pg)(x,\omega) - 2\pi i x\, V(f,g)(x,\omega).$$

(ii) *If f belongs to Dom(Q), then $V(f,g)$ has a continuous partial derivative with respect to the first argument x, and we have*

$$\frac{\partial}{\partial x}V(f,g)(x,\omega) = V(Qf,g)(x,\omega) - 2\pi i\omega\, V(f,g)(x,\omega).$$

If f (resp. g) belongs to Schwartz class $\mathcal{S}(\mathbb{R})$, then Pf, Qf (resp. Pg, Qg) $\in \mathcal{S}(\mathbb{R})$, as well. Iterated application of Proposition 4.4 or Corollary 4.5 gives the following smoothness result for the STFT.

Theorem 4.6. *Let $f, g \in L^2(\mathbb{R})$, and at least one of them in Schwartz class $\mathcal{S}(\mathbb{R})$. Then $V(f,g)(x,\omega)$ is infinitely partially differentiable in both variables x and ω.*

Although this result may be considered mathematical folklore, to our knowledge it has not been stated and proved in the literature so far. Note that this proves in particular that the STFT with Gaussian window is smooth.

4.2. **The Derivative of the Phase Around the Zeros of the STFT.** In this section we present an analytic explanation of the peculiar behaviour of the phase derivatives of the STFT for a large class of window functions. It turns out that the phenomenon is connected to the smoothness and continuous differentiability of the STFT which, as we have seen in the previous paragraph, is in turn connected to the smoothness of the window.

Consider first the partial derivative of the phase of the STFT with respect to the first variable (i.e. 'time'). For convergence along a vertical path, we have

Theorem 4.7 (Phase derivatives of the STFT, part I). *Let $f, g \in L^2(\mathbb{R})$. Assume that*

- $V(f,g) = V = U + i\cdot W \in C^2(\mathbb{R}^2)$
- $V(x_0,\omega_0) = 0$

- $\det J_V(x_0, \omega_0) < 0$, *where*

$$J_V(x_0,\omega_0) = \begin{pmatrix} \frac{\partial U}{\partial x}(x_0,\omega_0) & \frac{\partial U}{\partial \omega}(x_0,\omega_0) \\ \frac{\partial W}{\partial x}(x_0,\omega_0) & \frac{\partial W}{\partial \omega}(x_0,\omega_0) \end{pmatrix}$$

denotes the Jacobian matrix of V at the point (x_0, ω_0)

Then the phase $\psi(x,\omega)$ of $V(f,g)(x,\omega)$ satisfies

$$\lim_{\omega\to\omega_0} \frac{\partial \psi}{\partial x}(x_0,\omega) = \begin{cases} +\infty, & \text{if } \omega \uparrow \omega_0 \text{ from below} \\ -\infty, & \text{if } \omega \downarrow \omega_0 \text{ from above.} \end{cases}$$

Proof. We have

$$\frac{\partial \psi}{\partial x}(x_0,\omega) = \frac{U(x_0,\omega)\cdot W_x(x_0,\omega) - W(x_0,\omega)\cdot U_x(x_0,\omega)}{U^2(x_0,\omega)+W^2(x_0,\omega)}.$$

Since W_x and U_x are continuous and thus remain bounded in a neighbourhood of (x_0, ω_0), both numerator and denominator tend to zero for $\omega \to \omega_0$. However, both functions are differentiable, since $V \in C^2(\mathbb{R}^2, \mathbb{R}^2)$. So L'Hospital's Rule is applicable and yields

$$\begin{aligned}
&\lim_{\omega\to\omega_0} \frac{U(x_0,\omega)\cdot W_x(x_0,\omega) - W(x_0,\omega)\cdot U_x(x_0,\omega)}{U^2(x_0,\omega)+W^2(x_0,\omega)} \\
&\quad\overset{\text{L'Hosp.}}{=} \lim_{\omega\to\omega_0} \frac{\frac{d}{d\omega}\left(U(x_0,\omega)\cdot W_x(x_0,\omega) - W(x_0,\omega)\cdot U_x(x_0,\omega)\right)}{\frac{d}{d\omega}\left(U^2(x_0,\omega)+W^2(x_0,\omega)\right)} \\
&\quad= \lim_{\omega\to\omega_0} \frac{\left(U_\omega W_x + U W_{x\omega} - W_\omega U_x - W U_{x\omega}\right)(x_0,\omega)}{\left(2UU_\omega + 2WW_\omega\right)(x_0,\omega)} \\
&\quad= \lim_{\omega\to\omega_0} \frac{\left(U W_{x\omega} - W U_{x\omega}\right)(x_0,\omega) + \left(U_\omega W_x - W_\omega U_x\right)(x_0,\omega)}{\left(2UU_\omega + 2WW_\omega\right)(x_0,\omega)},
\end{aligned}$$

if the latter exists.
We clearly have

$$\left(U W_{x\omega} - W U_{x\omega}\right)(x_0,\omega) \to 0,$$

since $U(x_0,\omega) \to U(x_0,\omega_0) = 0$, $W(x_0,\omega) \to W(x_0,\omega_0) = 0$, and $W_{x\omega}$ and $U_{x\omega}$ are continuous and thus remain bounded. Furthermore,

$$\begin{aligned}
\left(U_\omega W_x - W_\omega U_x\right)(x_0,\omega) &\to \left(U_\omega W_x - W_\omega U_x\right)(x_0,\omega_0) \\
&= -\det J_V(x_0,\omega_0) \neq 0,
\end{aligned}$$

by assumption. Hence the numerator tends to a nonzero number, in this case ($\det J_V(x_0, \omega_0) < 0$) a positive one. For the denominator, we find

$$\begin{aligned}
\left(2UU_\omega + 2WW_\omega\right)(x_0,\omega) &= \frac{d}{d\omega}\left(U^2+W^2\right)(x_0,\omega)\\
&= \begin{cases} <0, & \text{if } \omega<\omega_0\\ >0, & \text{if } \omega>\omega_0\end{cases}
\end{aligned}$$

since the function $\omega \mapsto \left(2UU_\omega + 2WW_\omega\right)(x_0,\omega)$ has a strict local minimum in ω_0. At the same time,

$$\left(2UU_\omega + 2WW_\omega\right)(x_0,\omega) \to 0$$

for $\omega \to \omega_0$, hence the denominator goes to zero from below for $\omega \uparrow \omega_0$ and from above for $\omega \downarrow \omega_0$. The claim follows. $\square$

Note that for simplicity we consider only the case that the Jacobian determinant $\det J_V(x_0,\omega_0)$ is negative; this case corresponds to the examples we presented above. For positive Jacobian determinant, the situation is completely analogous, although reversed in the sense that the positive and negative singularities switch roles. Apart from this, the general behavior remains the same.

For convergence along a horizontal path, we need slightly more regularity:

Theorem 4.8 (Phase derivatives of the STFT, part II)**.** *Let $f,g \in L^2(\mathbb{R})$. Assume that*

- $V(f,g) = V = U + i\cdot W \in C^3(\mathbb{R}^2)$
- $V(x_0,\omega_0) = 0$
- $\det J_V(x_0,\omega_0) < 0$*, where J is the Jacobian as in Theorem 4.7*

Then the phase $\psi(x,\omega)$ of $V(f,g)(x,\omega)$ satisfies

$$\lim_{x\to x_0} \frac{\partial \psi}{\partial x}(x,\omega_0) = c \in \mathbb{R}, \quad \textit{if } x \to x_0,$$

i.e. it converges to some real number $c \in \mathbb{R}$.

Proof. The assumptions allow to apply L'Hospital's Rule twice, giving

$$\begin{aligned}
&\lim_{x\to x_0} \frac{\partial\psi}{\partial x}(x,\omega_0)\\
&\quad= \lim_{x\to x_0} \frac{U(x,\omega_0)\cdot W_x(x,\omega_0) - W(x,\omega_0)\cdot U_x(x,\omega_0)}{U^2(x,\omega_0)+W^2(x,\omega_0)}\\
&\quad\overset{\text{L'Hosp.}}{=} \lim_{x\to x_0} \frac{\frac{d}{dx}\left(U(x,\omega_0)\cdot W_x(x,\omega_0) - W(x,\omega_0)\cdot U_x(x,\omega_0)\right)}{\frac{d}{dx}\left(U^2(x,\omega_0)+W^2(x,\omega_0)\right)}\\
&\quad= \lim_{x\to x_0} \frac{\left(U_xW_x + UW_{xx} - W_xU_x - WU_{xx}\right)(x,\omega_0)}{\left(2UU_x+2WW_x\right)(x,\omega_0)}\\
&\quad= \lim_{x\to x_0} \frac{\left(UW_{xx} - WU_{xx}\right)(x,\omega_0)}{\left(2UU_x+2WW_x\right)(x,\omega_0)}\\
&\quad\overset{\text{L'Hosp.}}{=} \lim_{x\to x_0} \frac{\frac{d}{dx}\left(UW_{xx} - WU_{xx}\right)(x,\omega_0)}{\frac{d}{dx}\left(2UU_x+2WW_x\right)(x,\omega_0)}\\
&\quad= \lim_{x\to x_0} \frac{\left(U_xW_{xx} + UW_{xxx} - W_xU_{xx} - WU_{xxx}\right)(x,\omega_0)}{2\left(U_x^2+UU_{xx}+W_x^2+WW_{xx}\right)(x,\omega_0)},
\end{aligned}$$

if the latter limit exists. For the denominator,

$$\left(UU_{xx}+WW_{xx}\right)(x,\omega_0) \to \left(UU_{xx}+WW_{xx}\right)(x_0,\omega_0) = 0$$

for $x \to x_0$, but

$$\left(U_x^2+W_x^2\right)(x,\omega_0) \to \left(U_x^2+W_x^2\right)(x_0,\omega_0) > 0$$

converges to a nonzero number, since not both $U_x(x_0,\omega_0)$ and $W_x(x_0,\omega_0$ can be zero because of $\det J_V(x_0,\omega_0) = \left(U_xW_\omega - U_\omega W_x\right)(x_0,\omega_0) \neq 0$. The numerator obviously converges:

$$\begin{aligned}
&\left(U_xW_{xx} + UW_{xxx} - W_xU_{xx} - WU_{xxx}\right)(x,\omega_0)\\
&\qquad \to \left(U_xW_{xx} - W_xU_{xx}\right)(x_0,\omega_0) \in \mathbb{R},
\end{aligned}$$

thus

$$\lim_{x\to x_0} \frac{\partial\psi}{\partial x}(x,\omega_0) = \frac{\left(U_xW_{xx} - W_xU_{xx}\right)(x_0,\omega_0)}{2\left(U_x^2+W_x^2\right)(x_0,\omega_0)} =: c \in \mathbb{R}.$$

□

Concerning the partial derivatives of the phase of the STFT with respect to the second variable ('frequency'), we can argue almost identically and thus find the following analogous results:

Theorem 4.9 (Phase derivatives of the STFT, part III)**.** *Let $f, g \in L^2(\mathbb{R})$. Assume that*

- $V(x_0,\omega_0) = 0$
- $\det J_V(x_0,\omega_0) < 0$

Let $V(f,g) = V = U + i \cdot W \in C^2(\mathbb{R}^2)$*. Then the phase* $\psi(x,\omega)$ *of* $V(f,g)(x,\omega)$ *satisfies*

$$\lim_{x\to x_0} \frac{\partial \psi}{\partial \omega}(x,\omega_0) = \begin{cases} +\infty, & \text{if } x \to x_0 \text{ from the left} \\ -\infty, & \text{if } x_0 \leftarrow x \text{ from the right.} \end{cases}$$

Let $V(f,g) = V = U + i \cdot W \in C^3(\mathbb{R}^2)$*, then*

$$\lim_{\omega\to\omega_0} \frac{\partial \psi}{\partial \omega}(x_0,\omega) = c' \in \mathbb{R}, \quad \text{if } \omega \to \omega_0,$$

converges to some real number $c' \in \mathbb{R}$*.* □

Acknowledgements

The work on this paper was partly supported by the Austrian Science Fund (FWF) START-project FLAME ('Frames and Linear Operators for Acoustical Modeling and Parameter Estimation'; Y 551-N13). The authors are thankful to the project partners for fruitful discussions and valuable comments, in particular to B. Torrésani. Particular thanks go to P. Flandrin for his kind encouragement.

References

[1] F. Auger and P. Flandrin. Improving the readability of timefrequency and time-scale representationsby the method of re-assigment. *IEEE Trans. Signal Process.*, 43(5):1068–1089, May 1995.

[2] Franois Auger, ric Chassande-Mottin, and Patrick Flandrin. On phase-magnitude relationships in the short-time fourier transform. *IEEE Signal Process. Lett.*, 19(5):267–270, 2012.

[3] P. Balazs. Basic definition and properties of Bessel multipliers. *Journal of Mathematical Analysis and Applications*, 325(1):571–585, January 2007.

[4] Peter Balazs, Holger Waubke, and Werner A. Deutsch. Phasenanalyse mit akustischen Anwendungsbeispielen. In *Proceedings DAGA 2003 - Fortschritte der Akustik*, March 2003.

[5] R Carmona, W.-L. Hwang, and B. Torrésani. *Practical Time-Frequency Analysis*. Academic Press San Diego, 1998.

[6] N. Delprat, B. Escudie, P. Guillemain, R. Kronland-Martinet, P. Tchamitchian, and B. Torrésani. Asymptotic wavelet and Gabor analysis: extraction of instantaneous frequencies. *IEEE Transactions on Information Theory*, 38(2):644–664, 1992.

[7] M. Dolson. The phase vocoder: a tutorial. *Computer Musical Journal*, 10(4):11–27, 1986.

[8] F. Auger E. Chassande-Mottin and P. Flandrin. On the statistics of spectrogram reassignment vectors. *Multidim. Syst. and Signal Proc.*, 9(4):355–362, 1998.

[9] H. G. Feichtinger and K. Nowak. *A first survey of Gabor multipliers*, chapter 5, pages 99–128. Birkhäuser Boston, 2003.

[10] J.L. Flanagan and R. M. Golden. Phase vocoder. *Bell Syst. Tech.*, 45:1493 – 1509, 1966.

[11] P. Flandrin. *Time-Frequency/Time-Scale Analysis.* Academic Press, San Diego, 1999.

[12] Timothy Gardner and Marcelo Magnasco. Sparse time-frequency representations. *Proc. Natl. Acad. Sci. USA*, 103(16):6094–6099, April 2006.

[13] F. Jaillet, P. Balazs, M. Dörfler, and N. Engelputzeder. On the structure of the phase around the zeros of the short-time fourier transform. In Marinus M. Boone, editor, *NAG/DAGA 2009*, pages 1584–1587, Rotterdam, 2009.

[14] K. Kodera, C. De Villedary, and R. Gendrin. A new method for the numerical analysis of nonstationary signals(magnetospheric ULF). *Physics of the Earth and Planetary Interiors*, 12(2):142–150, 1976.

[15] A. V. Oppenheim and J. S. Lim. The importance of phase in signals. *Proceedings of the IEEE*, 69(5):529–541, May 1981.

[16] N. Perraudin, P. Balazs, and P. Soendergaard. A fast Griffin-Lim algorithm. In *2013 IEEE Workshop on Applications of Signal Processing to Audio and Acoustics, New Paltz, NY, USA, 2013.*, 2013.

[17] Thomas Quatieri. *Discrete-time speech signal processing: principles and practice.* Prentice Hall Press, Upper Saddle River, NJ, USA, 2001.

[18] Diana T. Stoeva and Peter Balazs. Invertibility of multipliers. *Applied and Computational Harmonic Analysis*, 33(2):292–299, 2012.

[19] Diana T. Stoeva and Peter Balazs. Canonical forms of unconditionally convergent multipliers. *Journal of Mathematical Analysis and Applications*, 399:252–259, 2013.