Atomic decompositions of square-integrable functions

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Abstract

This report serves as a survey for the discrete expansion of square-integrable functions of one real variable on an interval $I \subseteq \mathbb{R}$. The framework for the expansions used in this report are frames for the separable Hilbert space of square-integrable functions on I. After an overview of atomic decompositions of Banach spaces and frames for separable Hilbert spaces in general, some concrete examples of frames are described, namely the Fourier frame, the Gabor frame and the wavelet frame. The description of these specific frames consists of the admissible and sufficient conditions, operators associated with the frames and the construction of the associated canonical dual frame.

1 Introduction

A decomposition of a function is a representation of a function in terms of elementary functions. The representation of a function in terms of elementary functions is in general called an *expansion* of a function. A discrete expansion of an arbitrary function f is a representation of f in terms of a *series expansion* of the form

$$f = \sum_{n \in J} c_n f_n,$$

where J denotes a discrete, countable index set. The functions f_n, $n \in J$, are in this case the elementary functions and are called the *expansion functions*. These expansion functions are in general derived by elementary operations on a single, fixed function called the *generator* or *atom*. A collection of expansion functions $\{f_n\}_{n \in J}$ that is used to represent a function as a convergent series is called a *system*. The scalars c_n, $n \in J$, that are associated with the expansion functions are in general called the *coefficients*. The collection of coefficients $\{c_n\}_{n \in J}$ form a representation of the expanded function in a domain that is in general called the *transform domain* associated with the expansion functions. The process of mapping an arbitrary function into the transform domain associated with the expansion functions, is in general called *analysis*. The mapping that maps a function into a collection of coefficients in the transform domain is called the *coefficient mapping*. The process that maps a collection of coefficients from the transform domain into an arbitrary function is in general called *synthesis*. The mapping that constructs a function from a collection of coefficients in the transform domain is called the *reconstruction mapping*. Both the analysis and synthesis of functions in terms of a linear combination of expansion functions belong to the general framework of an atomic decomposition.

2 Atomic decompositions

The concept of representing an arbitrary function as a series expansion with respect to a system of expansion functions can be generalized to classes of functions. In this case can *each* element of the class of functions be represented as a series expansion with respect to a system of expansion functions. An example of a general method for such decompositions is an *atomic decomposition*. An atomic decomposition can be defined for general Banach spaces and includes therefore the space of square-integrable functions, the space that is considered in this paper, as a special case.

Definition 1 (Atomic decomposition)**.** Let $\mathcal{B}$ be a Banach space and let $\mathcal{B}_d$ be an associated Banach space of scalar-valued sequences indexed by elements of the countable index set $J \subseteq \mathbb{Z}$. Let $\{f_n\}_{n\in J} \subset \mathcal{B}^*$ and $\{g_n\}_{n\in J} \subset \mathcal{B}$. Then the pair of systems $(\{f_n\}_{n\in J}, \{g_n\}_{n\in J})$ is called an *atomic decomposition* of $\mathcal{B}$ with respect to $\mathcal{B}_d$ if it satisfies the following conditions:

(i) $\{\langle f, f_n\rangle\} \in \mathcal{B}_d, \quad \forall f \in \mathcal{B}$,

(ii) $\exists 0 < A, B < \infty,\ \forall f \in \mathcal{B},\ A\|f\|_{\mathcal{B}} \le \|\langle f, f_n\rangle\|_{\mathcal{B}^d} \le B\|f\|_{\mathcal{B}}$,

(iii) $f = \sum_{n\in J}\langle f, f_n\rangle g_n \quad \forall f \in \mathcal{B}$.

The constants A and B that appear in (ii) called *atomic bounds* of the atomic decomposition $(\{f_n\}_{n\in J}, \{g_n\}_{n\in J})$.

An atomic decomposition allows that any function $f \in \mathcal{B}$ can be represented as

$$f = \sum_{n\in J}\langle f, f_n\rangle g_n. \tag{1}$$

This formula is in general called the *reproducing formula* for $f \in \mathcal{B}$. If the series of the reproducing formula associated with an atomic decomposition converges unconditionally, then the atomic decomposition is called *unconditional*. The collection of scalars $\{\langle f, f_n\rangle\}_{n\in J}$ in the reproducing formula are obtained through the coefficient mapping defined by

$$f \mapsto \{\langle f, f_n\rangle\}. \tag{2}$$

The definition of this coefficient mapping is allowed by conditions (i) and (ii) of definition 1. If the Banach spaces $\mathcal{B}$ and $\mathcal{B}_d$ of definition 1 are respectively the separable Hilbert space $\mathcal{H}$ and $\ell^2(J)$, then conditions (i) and (ii) of definition 1 imply the existence of a system $\{g_n\}_{n\in J} \subset \mathcal{H}$ that forms together with the system $\{f_n\}_{n\in J} \subset \mathcal{H}$ an atomic decomposition of $\mathcal{H}$ with respect to $\ell^2(J)$. Thus, if the norms $\|f\|_{\mathcal{H}}$ and $\|\langle f, f_n\rangle\|_{\ell^2}$ are equivalent, then the reproducing formula in (1) exists for all $f \in \mathcal{H}$. The system $\{f_n\}_{n\in J} \subset \mathcal{H}$ is in this case called a *discrete frame* for $\mathcal{H}$.

2.1 Frame theory

A system $\{f_n\}_{n\in J} \subset \mathcal{H}$ is a frame for the Hilbert space $\mathcal{H}$ if it satisfies the so-called *frame condition.*

Definition 2 (Frame condition)**.** A system $\{f_n\}_{n\in J} \subseteq \mathcal{H}$ is called a *discrete frame* for a separable Hilbert space $\mathcal{H}$ if there exist two constants $A, B > 0$ such that

$$A\,||f||^2_{\mathcal{H}} \;\le\; \sum_{n\in J}|\langle f, f_n\rangle|^2 \;\le\; B\,||f||^2_{\mathcal{H}}\,, \quad \forall f \in \mathcal{H}. \tag{3}$$

The constants A and B are called the lower respectively the upper frame bound of the frame $\{f_n\}_{n\in J}$.

If a system $\{f_n\}_{n\in J}$ is a frame for $\mathcal{H}$, then $\overline{\text{span}}\{f_n\}_{n\in J} = \mathcal{H}$. This shows that if a system is a frame for a seperable Hilbert space, then it is *complete* in this Hilbert space.The elements f_n, $n \in J$, of a frame $\{f_n\}_{n\in J}$ are called the *frame elements* of $\{f_n\}_{n\in J}$. If $\{f_n\}_{n\in J}$ is a frame, but ceases to be a frame after the removal of any of its frame elements, then $\{f_n\}_{n\in J}$ is called an *exact frame.* If the frame $\{f_n\}_{n\in J}$ is still a frame after the removal of any of its frame elements, then it is called an *overcomplete frame.*

The frame bounds of a frame are inseparable related to the possibility of representing a $f \in \mathcal{H}$ in terms of the frame elements as in the reproducing formula (1). However, the frame bounds of a frame are in general not unique. An example of two important frame bounds are the so-called *optimal frame bounds.*

Definition 3. The largest lower frame bound and the smallest upper frame bound are called the *optimal frame bounds*. The optimal lower frame bound A respectively the optimal upper frame bound B are defined as

$$A := \inf_{f \in \mathcal{H}} \left(\frac{\sum_{n \in J} |\langle f, f_n \rangle|^2}{\|f\|_{\mathcal{H}}^2} \right),$$
$$B := \sup_{f \in \mathcal{H}} \left(\frac{\sum_{n \in J} |\langle f, f_n \rangle|^2}{\|f\|_{\mathcal{H}}^2} \right).$$

If the frame bounds of a frame are equal, then the frame is called *tight*. For a tight frame $\{f_n\}_{n \in J}$ whose frame bounds are equal to 1, the frame condition becomes Parseval's equality for orthonormal bases,

$$\sum_{n \in J} |\langle f, f_n \rangle|^2 = ||f||_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}. \tag{4}$$

From this it can be deduced that a orthonormal basis is a tight frame with frame bound 1. It can further be deduced that if a system $\{f_n\}_{n \in J}$ only satisfies the upper bound in the frame condition, then the frame condition becomes *Bessel's inequality*,

$$\sum_{n \in J} |\langle f, f_n \rangle|^2 \leq B \, ||f||_{\mathcal{H}}^2, \quad \forall f \in \mathcal{H}. \tag{5}$$

In this case is the system $\{f_n\}_{n \in J}$ called a *Bessel sequence* with Bessel bound B.

2.1.1 Operators associated with frames

There are a few important operators associated with frames, including the coefficient operator, the representation operator and the frame operator.

Definition 4. Let $\{f_n\}_{n \in J} \subseteq \mathcal{H}$ be a frame for a Hilbert space $\mathcal{H}$.

(i) The *coefficient operator* $\mathcal{C}$ associated with $\{f_n\}_{n \in J}$ is the mapping defined by

$$\mathcal{C} : \mathcal{H} \to \ell^2(J), \quad f \mapsto \mathcal{C}f = \{\langle f, f_n \rangle\}_{n \in J}.$$

(ii) The *representation operator* $\mathcal{R}$ associated with $\{f_n\}_{n \in J}$ is the mapping defined by

$$\mathcal{R} : \ell^2(I) \to \mathcal{H}, \quad f \mapsto \mathcal{R}c = \sum_{n \in J} c_n f_n.$$

(iii) The *frame operator* $\mathcal{S}$ associated with $\{f_n\}_{n \in J}$ is the mapping defined by

$$\mathcal{S} : \mathcal{H} \to \mathcal{H}, \quad f \mapsto \mathcal{S}f = \sum_{n \in J} \langle f, f_n \rangle f_n.$$

Several important properties of the operators defined above are given in the following proposition.

Proposition 1. *Let $\{f_n\}_{n \in J} \subseteq \mathcal{H}$ be a frame for a Hilbert space $\mathcal{H}$.*

(i) The coefficient operator $\mathcal{C}$ and the representation operator $\mathcal{R}$ are both bounded operators.

(ii) The representation operator $\mathcal{R}$ is the adjoint operator of the coefficient operator $\mathcal{C}$.

(iii) The frame operator $\mathcal{S}$ is bounded, self-adjoint, positive and invertible.

Proof. (i) Since the frame $\{f_n\}_{n\in J}$ associated with the coefficient operator $\mathcal{C}$ is a Bessel sequence,

$$\|\mathcal{C}f\|_{\ell^2} = \sum_{n\in J} |\ \langle f, f_n\rangle\ |^2\ \le B\,||f||^2_{\mathcal{H}}\,,$$

it can immediately be deduced that $\mathcal{C}$ is bounded. The representation operator $\mathcal{R}$ is bounded since it is the adjoint of $\mathcal{C}$ (see (ii)).

(ii) Let $(c_n)_{n\in J} \subseteq \ell^2(J)$. Then

$$\langle c, \mathcal{C}f\rangle = \sum_{n\in J} c_n\overline{\langle f, f_n\rangle} = \left\langle \sum_{n\in J} c_n f_n, f\right\rangle = \langle \mathcal{R}c, f\rangle.$$

(iii) Since $\mathcal{S} = \mathcal{RC}$, $\mathcal{S}$ is the composition of two bounded operators and is itself also bounded since

$$\|\mathcal{S}f\| = \|\mathcal{RC}f\| \le \|\mathcal{R}\|\|\mathcal{C}f\| \le \|\mathcal{R}\|\|\mathcal{C}\|\|f\|, \quad \forall f \in \mathcal{H}.$$

That $\mathcal{S}$ is self-adjoint follows from the direct computation

$$\langle \mathcal{S}f, f\rangle = \sum_{n\in J} |\langle f, f_n\rangle|^2 = \langle f, \mathcal{S}f\rangle.$$

That $\mathcal{S}$ is positive follows directly from the frame condition rewritten in the form

$$A\|f\|^2_{\mathcal{H}} \le \langle \mathcal{S}f, f\rangle \le B\|f\|^2_{\mathcal{H}}$$

with $A, B > 0$. Since $A > 0$ and $\mathcal{S}f = 0 \Longrightarrow f = 0$, $\mathcal{S}$ is invertible.

□

2.2 Frame decompositions

The invertibility of the frame operator $\mathcal{S}$ leads to the construction of the system $\{\mathcal{S}^{-1}f_n\}_{n\in J}$. In the following lemma it is proved that this system constitutes a frame, called the *canonical dual* of the frame $\{f_n\}_{n\in J} \subseteq \mathcal{H}$.

Lemma 1. *If $\{f_n\}_{n\in J} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ with frame bounds A, B and if $\mathcal{S}$ is the frame operator associated with $\{f_n\}_{n\in J}$, then is $\{\mathcal{S}^{-1}f_n\}_{n\in J}$ a frame with frame bounds B^{-1}, A^{-1}.*

Proof. Since $\mathcal{S}$ is a positive operator it holds that

$$A\|f\|^2_{\mathcal{H}} \le \langle \mathcal{S}f, f\rangle \le B\|f\|^2_{\mathcal{H}} \iff A\|f\|_{\mathcal{H}} \le \|\mathcal{S}f\| \le B\|f\|_{\mathcal{H}}.$$

Let now $f = \mathcal{S}^{-1}g$ where $g \in \mathcal{H}$. Then

$$\begin{aligned} A\|f\|_{\mathcal{H}} \le \|\mathcal{S}f\| \le B\|f\|_{\mathcal{H}} &\iff B^{-1}\|\mathcal{S}f\| \le \|f\| \le A^{-1}\|\mathcal{S}f\| \\ &\iff B^{-1}\|g\| \le \|\mathcal{S}^{-1}g\| \le A^{-1}\|g\| \\ &\iff B^{-1}\|g\|^2 \le \langle \mathcal{S}^{-1}g, g\rangle \le A^{-1}\|g\|^2 \\ &\iff B^{-1}\|g\|^2 \le \sum_{n\in J} |\langle g, \mathcal{S}^{-1}g\rangle|^2 \le A^{-1}\|g\|^2 \end{aligned}$$

□

Theorem 1 (Frame decomposition)**.** *If a system $\{f_n\}_{n\in J}$ forms a frame for a Hilbert space $\mathcal{H}$ and $\mathcal{S}$ is the frame operator associated with this frame, then any $f \in \mathcal{H}$ can be represented as the series*

$$f = \sum_{n\in J} \langle f, \mathcal{S}^{-1}f_n\rangle f_n \tag{6}$$

$$f = \sum_{n\in J} \langle f, f_n\rangle \mathcal{S}^{-1} f_n \tag{7}$$

where both series converges unconditionally in the norm of $\mathcal{H}$.

Proof. Since the frame operator $\mathcal{S}$ associated with a frame $\{f_n\}_{n\in J}$ is invertible and self-adjoint, it follows from direct computations that

$$f = \mathcal{S}\mathcal{S}^{-1} f = \sum_{n\in J} \langle \mathcal{S}^{-1} f, f_n\rangle f_n = \sum_{n\in J} \langle f, \mathcal{S}^{-1} f_n\rangle f_n$$

$$f = \mathcal{S}^{-1}\mathcal{S} f = \mathcal{S}^{-1} \sum_{n\in J} \langle f, f_n\rangle f_n = \sum_{n\in J} \langle f, f_n\rangle \mathcal{S}^{-1} f_n$$

From the fact that both $\{f_n\}_{n\in J}$ and $\{\mathcal{S}^{-1} f_n\}_{n\in J}$ are Bessel sequences and both $\{\langle \mathcal{S}^{-1} f, f_n\rangle\}_{n\in J}$ and $\{\langle f, f_n\rangle\}_{n\in J}$ are in $\ell^2(J)$, it can be deduced that the series (6) respectively (7) converge unconditionally. For the unconditional convergence of Bessel sequences, see [7]. □

The frame decomposition allows an atomic decomposition of $\mathcal{H}$ with respect to $\ell^2(J)$. Furthermore, since the frame decomposition implies unconditional convergence of the series, the atomic decomposition that is connected to the frame decomposition is an unconditional atomic decomposition. The atomic decompositions are in this case the pairs of systems $(\{f_n\}_{n\in J}, \{\mathcal{S}^{-1} f_n\}_{n\in J})$ and $(\{\mathcal{S}^{-1} f_n\}_{n\in J}, \{f_n\}_{n\in J})$ which consist of a frame and its canonical dual.

Remark 1. Two frames that satisfy the reproducing formula in (1) are in general called *dual frames.*

Theorem 2. *Let $\{f_n\}_{n\in J}$ and $\{g_n\}_{n\in J}$ be Bessel sequences in $\mathcal{H}$. If $\{f_n\}_{n\in J}$ and $\{g_n\}_{n\in J}$ satisfy the equality*

$$\langle f, g\rangle = \sum_{n\in J} \langle f, f_n\rangle \langle g_n, g\rangle, \quad \forall f, g \in \mathcal{H}. \tag{8}$$

Then $\{f_n\}_{n\in J}$ and $\{g_n\}_{n\in J}$ are dual frames for $\mathcal{H}$.

Proof. That a Bessel sequences $\{f_n\}_{n\in J}$ that satisfies (8) is a frame follows from the computation

$$\|f\|_{\mathcal{H}}^2 = \langle f, f\rangle = \sum_{n\in J} \langle f, f_n\rangle \langle g_n, f\rangle \le \sqrt{\sum_{n\in J} |\langle f, f_n\rangle|^2} \sqrt{\sum_{n\in J} |\langle f, g_n\rangle|^2} \le \sqrt{\sum_{n\in J} |\langle f, f_n\rangle|^2} \sqrt{B} \|f\|_{\mathcal{H}},$$

which is equivalent to

$$B^{-1} \|f\|_{\mathcal{H}}^2 \le \sum_{n\in J} |\langle f, f_n\rangle|^2.$$

In the same way it can be shown that the Bessel sequence $\{g_n\}_{n\in J}$ satisfies the lower frame bound B^{-1} through which it becomes a frame. That the frames $\{f_n\}_{n\in J}$ and $\{g_n\}_{n\in J}$ are dual frames follows from

$$0 = \langle f, g\rangle - \sum_{n\in J} \langle f, f_n\rangle \langle g_n, g\rangle = \left\langle f - \sum_{n\in J} \langle f, f_n\rangle g_n, g \right\rangle, \quad \forall g \in \mathcal{H},$$

which shows that $f = \sum_{n\in J} \langle f, f_n\rangle g_n$. □

A dual frame of a frame that isn't the canonical dual is called an *alternative dual* of the frame. The existence of alternative dual frames is related to the completeness of the frame.

Theorem 3. *Let $\{f_n\}_{n\in J}$ be an overcomplete frame. Then there exists a frame $\{g_n\}_{n\in J} \neq \{\mathcal{S}^{-1} f_n\}_{n\in J}$ that satisfy*

$$f = \sum_{n\in J} \langle f, g_n\rangle f_n, \quad \forall f \in \mathcal{H}.$$

Proof. See [2]. □

3 Fourier expansion

The theory of a Fourier expansion is based on the notion that any periodic function can be decomposed into a superposition of sinusoids. A function f on $\mathbb{R}$ is called a $\mathbb{Z}$-periodic or 1-periodic function if

$$f(t) = f(t+p), \quad \forall p \in \mathbb{Z}. \tag{9}$$

These functions can be represented as functions on the closed interval $[0,1]$ or as functions on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and are therefore elements of $L^2[0,1]$ or $L^2(\mathbb{T})$. An example of an element of $L^2(\mathbb{T})$ is the complex exponential $e^{i2\pi kt}$ with $k \in \mathbb{Z}$ and $t \in \mathbb{T}$. A collection of such complex exponentials given by

$$\{e_k\}_{k\in\mathbb{Z}} = \{e^{i2\pi kt}\}_{k\in\mathbb{Z}}, \quad t \in \mathbb{T} \tag{10}$$

forms a Parseval frame for $L^2(\mathbb{T})$ [7] .

3.1 Operators associated with Fourier expansions

The coefficient operator associated with the frame $\{e_k\}_{k\in\mathbb{Z}}$ is given by

$$\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}), \quad \mathcal{F}f = \{\langle f, e_k\rangle\}_{k\in\mathbb{Z}} \tag{11}$$

The coefficient operator $\mathcal{F}$ maps a function f from the so-called *time domain* into the *frequency domain*. The collection of scalars $\{\langle f, e_k\rangle\}_{k\in\mathbb{Z}}$ in the frequency domain are called the *Fourier coefficients* of f. The adjoint of the coefficient operator operator $\mathcal{F}$, $\mathcal{F}^*$, is given by

$$\mathcal{F}^* : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T}), \quad \mathcal{F}^*c = \sum_{k\in\mathbb{Z}} c_k e_k \tag{12}$$

and is called the representation operator associated with the frame $\{e_k\}_{k\in\mathbb{Z}}$. This representation operator $\mathcal{F}^*$ maps a collection of scalars $\{c_k\}_{k\in\mathbb{Z}} \subset \ell^2(\mathbb{Z})$ into a function $f \in L^2(\mathbb{Z})$. The concatenation of $\mathcal{F}^*$ and $\mathcal{F}$, that is, $\mathcal{F}^*\mathcal{F}$, is the frame operator associated with the frame $\{e_k\}_{k\in\mathbb{Z}}$ and is given by

$$\mathcal{F}^*\mathcal{F}f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k \tag{13}$$

The series in (13) is in general called a *Fourier series.*

Theorem 4. *Let $\{e_k\}_{k\in\mathbb{Z}} = \{e^{i2\pi kt}\}_{k\in\mathbb{Z}}$ with $t \in \mathbb{T}$, then any $f \in L^2(\mathbb{T})$ can be represented as*

$$f = \sum_{k\in\mathbb{Z}} \langle f, e_k\rangle e_k$$

where the Fourier series converges unconditionally in the L^2-norm.

Proof. Since $\{e_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence and since $\{\langle f, e_k\rangle\}_{k\in\mathbb{Z}} \subset \ell^2(\mathbb{Z})$, the Fourier series defined through the frame operator associated with $\{e_k\}_{k\in\mathbb{Z}}$ converges unconditionally. □

4 Gabor expansion

Through the frame $\{e^{i2\pi kt}\}_{k\in\mathbb{Z}}$ any $f \in L^2(\mathbb{T})$ could be decomposed into a linear combination of complex exponentials with associated coefficients. A sequence of such frames can describe any square-integrable function on the interval $(-\infty, \infty)$, that is, any $f \in L^2(\mathbb{R})$.

4.1 Gabor frames

A Gabor expansion of a function $f \in L^2(\mathbb{R})$ into a sequence of linear combinations of complex exponentials with associated coefficients is based on the translation and modulation of a *generator*. Under certain conditions any $g \in L^2(\mathbb{R})$ could be such a generator, but a generator is admissible if it belongs to the so-called *Segal algebra* or *Feichtinger algebra* $S_0(\mathbb{R})$.

Definition 5 (Feichtinger algebra)**.** The function space $S_0(\mathbb{R})$ defined as

$$S_0(\mathbb{R}) = \left\{ g \in L^2(\mathbb{R}) \;\middle|\; \iint_{\mathbb{R}} \left| \int_{\mathbb{R}} g\,(t)\, e^{-\pi(t-\tau)^2} e^{-i2\pi\xi t} dt \right| d\tau d\xi < \infty \right\} \tag{14}$$

is called the *Feichtinger algebra* $S_0(\mathbb{R})$.

Remark 2. The Feichtinger algebra $S_0(\mathbb{R})$ is a Banach space with respect to the norm

$$\|g\|_{S_0} = \iint_{\mathbb{R}} \left| \int_{\mathbb{R}} g\,(t)\, e^{-\pi(t-\tau)^2} e^{-i2\pi\xi t} dt \right| d\tau d\xi, \tag{15}$$

and is a dense subspace in $L^2(\mathbb{R})$ [5].

Translated and modulated versions of a generator $g \in S_0(\mathbb{R})$ are obtained through the translation and modulation operator on $L^2(\mathbb{R})$.

Definition 6. Let $g \in L^2(\mathbb{R})$.

(i) The translation operator $\mathcal{T}_\tau$, that translates a function by $\tau \in \mathbb{R}$, is given by

$$\mathcal{T}_\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (\mathcal{T}_\tau g)(t) = g(t-\tau), \quad \tau \in \mathbb{R}.$$

(ii) The modulation operator $\mathcal{E}_\xi$, that translated a function by $\xi \in \mathbb{R}$, is given by

$$\mathcal{E}_\xi : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (\mathcal{E}_\xi g)(t) = g(t)e^{i2\pi\xi t}, \quad \xi \in \mathbb{R}.$$

A fundamental property of the translation and modulation operators, their so-called *commutation relation*, is proved in the next lemma.

Lemma 2. *For all $\tau, \xi \in \mathbb{R}$ the following hold:*

$$(\mathcal{T}_\tau \mathcal{E}_\xi g)(t) = e^{-i2\pi\xi t}(\mathcal{E}_\xi \mathcal{T}_\tau g)(t). \tag{16}$$

Proof. The relation follows from the direct computation,

$$\begin{aligned} (\mathcal{T}_\tau \mathcal{E}_\xi g)(t) &= \mathcal{E}_\xi g(t-\tau) \\ &= e^{i2\pi\xi(t-\tau)} g(t-\tau) \\ &= e^{-i2\pi\xi\tau} e^{i2\pi\xi t} g(t-\tau) \\ &=^{-i2\pi\xi\tau}\ \mathcal{E}_\xi \mathcal{T}_\tau g(t) \end{aligned}$$

□

Next, a version of a translated and modulated generator $g \in L^2(\mathbb{R})$ is given by

$$(\mathcal{E}_\xi \mathcal{T}_\tau g)(t) = g(t-\tau)e^{i2\pi\xi t}. \tag{17}$$

A natural discretization of such a translated and modulated generator is $\tau = n\alpha$, $\xi = m\beta$ where $\alpha, \beta > 0$ are fixed and $n, m \in \mathbb{Z}$ [3]. Such a subsampled version of a translated and modulated generator is called a *Gabor atom*.

Definition 7. Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}^+$. The collection of functions given by

$$\mathcal{G}(g, \alpha, \beta) = \{g_{m,n}\}_{m,n\in\mathbb{Z}}, \quad g_{m,n} := \mathcal{E}_{m\beta} \mathcal{T}_{n\alpha} g,$$

is called a *Gabor system*.

Definition 8. A Gabor system $\mathcal{G}(g, \alpha, \beta)$ that satisfies the frame condition (3) is called a *Gabor frame*.

4.1.1 Conditions and properties for Gabor frames

There are several sufficient conditions for a Gabor system to form a Gabor frame for $L^2(\mathbb{R})$. One of these sufficient conditions is stated in [1].

Theorem 5. *Let $g \in L^2(\mathbb{R})$ and $\alpha, \beta \in \mathbb{R}^+$. Suppose that*

$$A := \inf_{t\in[0,\alpha]} \left(\sum_{n\in\mathbb{Z}} |g(t-n\alpha)|^2 - \sum_{n\neq 0} |\sum_{n\in\mathbb{Z}} g(t-n\alpha)\overline{g(t-n\alpha-\frac{k}{\beta})}| \right) > 0,$$
$$B := \sup_{t\in[0,\alpha]} \sum_{n\in\mathbb{Z}} |\sum_{n\in\mathbb{Z}} g(t-n\alpha)\overline{g(t-n\alpha-\frac{k}{\beta})}| < \infty.$$

Then $\{\mathcal{E}_{mb}\mathcal{T}_{na}g\}_{m,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{\beta}$ and $\frac{B}{\beta}$.

Proof. See [1]. □

The completeness of a Gabor system $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ with $\alpha, \beta \in \mathbb{R}^+$ is related to the density of the lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and is therefore determined by the product of the parameters α, β.

Theorem 6. *Let $\mathcal{G}(g, \alpha, \beta)$ be a Gabor system. Then the following hold:*

(i) If $\alpha\beta \leq 1$, then $\mathcal{G}(g, \alpha, \beta)$ is overcomplete in $L^2(\mathbb{R})$,

(ii) If $\alpha\beta = 1$, then $\mathcal{G}(g, \alpha, \beta)$ is complete in $L^2(\mathbb{R})$,

(iii) If $\alpha\beta > 1$, then $\mathcal{G}(g, \alpha, \beta)$ is incomplete in $L^2(\mathbb{R})$.

Proof. See [6] □

4.2 Operators associated with Gabor frames

In this section the Gabor system $\mathcal{G}(g, \alpha, \beta)$ is assumed to form a Gabor frame for $L^2(\mathbb{R})$.

The coefficient operator $\mathcal{C}$ associated with $\mathcal{G}(g, \alpha, \beta)$ is explicitly given by

$$\mathcal{C} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2), \quad \mathcal{G}f = \{\langle f, \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\rangle\}_{m,n\in\mathbb{Z}}. \tag{18}$$

This coefficient operator maps a function f into a collection of scalars in the so-called *time-frequency domain*. The scalars $\{\langle f, \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\rangle\}_{m,n\in\mathbb{Z}}$ are called the *Gabor coefficients* associated with $\mathcal{G}(g, \alpha, \beta)$ and are points on the lattice $\Lambda := \alpha\mathbb{Z} \times \beta\mathbb{Z}$.

The adjoint of the coefficient operator $\mathcal{C}$ associated with $\mathcal{G}(g, \alpha, \beta)$ is the reconstruction operator $\mathcal{R}$ associated $\mathcal{G}(g, \alpha, \beta)$. This reconstruction operator is given by

$$\mathcal{R} : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad \mathcal{R} = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_{m,n}\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g. \tag{19}$$

This reconstruction operator expands a function in terms of a collection of scalars in the time-frequency domain. The concatenation of $\mathcal{C}$ and $\mathcal{R}$, that is, $\mathcal{R}^*\mathcal{C}$, is the frame operator $\mathcal{S}$ associated with the $\mathcal{G}(g, \alpha, \beta)$ and is called the *Gabor frame operator*. The Gabor frame operator is given by

$$\mathcal{S} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathcal{S}f = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\rangle \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g \tag{20}$$

An important property of the Gabor frame operator is its commutation with the operators $\mathcal{E}_{m\beta}$ and $\mathcal{T}_{n\alpha}$.

Lemma 3. *Let $\mathcal{G}(g, \alpha, \beta)$ be a Gabor frame and let $\mathcal{S}$ be the associated frame operator, then*

(i)

$$\mathcal{S}\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha} = \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\mathcal{S}, \quad \forall m, n \in \mathbb{Z}.$$

(ii)

$$\mathcal{S}^{-1}\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha} = \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\mathcal{S}^{-1}, \quad \forall m, n \in \mathbb{Z}.$$

Proof. (i) From a direct computation, using the commutator relation expressed in (16), it can be deduced that for any $f \in L^2(\mathbb{R})$,

$$\begin{aligned}
\mathcal{S}\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}f &= \\
&= \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} \langle \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}f, \mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g\rangle \mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g \\
&= \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} \langle f, \mathcal{T}_{-n\alpha}\mathcal{E}_{(m'-m)\beta}\mathcal{T}_{n'\alpha}g\rangle \mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g \\
&= \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} \langle f, e^{2\pi i n\alpha(m'-m)\beta}\mathcal{E}_{(m'-m)\beta}\mathcal{T}_{(n'-n)\alpha}g\rangle \mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g \\
&= \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} e^{2\pi i n\alpha m'\beta}\langle f, \mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g\rangle \mathcal{E}_{(m'-m)\beta}\mathcal{T}_{(n'-n)\alpha}g, \quad m'+m := m',\ n'+n := n' \\
&= \sum_{m'\in\mathbb{Z}}\sum_{n'\in\mathbb{Z}} e^{2\pi i n\alpha m'\beta}\langle f, \mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g\rangle \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\mathcal{E}_{m'\beta}\mathcal{T}_{n'\alpha}g \\
&= \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\mathcal{S}
\end{aligned}$$

(ii) Since $\mathcal{S}$ is a homeomorphishm, it can be replaced by $\mathcal{S}^{-1}$ in (i). □

Due to the commutativity of the inverse of the Gabor frame operator and the translation and modulation operator, the canonical dual of a Gabor frame $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ is explicitly given by

$$\mathcal{S}^{-1}\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g = \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\mathcal{S}^{-1}g, \tag{21}$$

where $\gamma = \mathcal{S}^{-1}g$ is called the *canonical dual generator*. This shows that the canonical dual of a Gabor frame is again a Gabor system, namely the Gabor frame $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\gamma\}_{m,n\in\mathbb{Z}}$.

Theorem 7. *If a Gabor system $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$, then there exists a Gabor frame for $L^2(\mathbb{R})$ given by $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\gamma\}_{m,n\in\mathbb{Z}}$ such that any $f \in L^2(\mathbb{R})$ can be represented as the series*

$$f = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle f, \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\rangle \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\gamma \tag{22}$$

$$f = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle f, \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\gamma\rangle \mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g \tag{23}$$

where both series converges unconditionally in the L^2-norm.

Proof. Since, by assumption, the system $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\}_{m,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ its canonical dual is given by $\{\mathcal{S}^{-1}\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}g\}_{m,n\in\mathbb{Z}}$. From lemma 3 it can be deduced that this canonical dual frame has the form $\{\mathcal{E}_{m\beta}\mathcal{T}_{n\alpha}\gamma\}_{m,n\in\mathbb{Z}}$ where $\gamma = \mathcal{S}^{-1}g$. Since both (22) and (23) are frame decompositions their unconditional convergence follows from theorem 1. □

This theorem provides two unconditional atomic decompositions of $L^2(\mathbb{R})$ with respect to $\ell^2(\mathbb{Z}^2)$, namely the ones given that consist of the Gabor frames $\mathcal{G}(g, \alpha, \beta)$ and $\mathcal{G}(\mathcal{S}^{-1}g, \alpha, \beta)$. In general, any two Gabor systems $\mathcal{G}(g, \alpha, \beta)$ and $\mathcal{G}(h, \alpha, \beta)$ form an atomic decomposition if they are dual frames of each other. In this case are the generators $g, h \in L^2(\mathbb{R})$ called *dual generators*, and is it possible to represent any $f \in L^2(\mathbb{R})$ as the series (22) and (23). Two Gabor frames are dual frames if they satisfy the general general duality condition for frames, but another condition for two Gabor systems to be dual frames is the so-called *Wexler-Raz duality condition.*

Theorem 8 (Wexler-Raz)**.** *Let $\mathcal{G}(g,\alpha,\beta)$ and $\mathcal{G}(h,\alpha,\beta)$ be two Gabor frames. The systems are dual frames if and only if*

$$\frac{\langle h, \mathcal{T}_{\frac{n}{\alpha}}\mathcal{E}_{\frac{m}{\beta}} g\rangle}{\alpha\beta} = \delta_{n,0}\delta_{m,0}, \quad m,n\in\mathbb{Z}. \tag{24}$$

Proof. See [2]. □

Any two Gabor frames that satisfy (24) form an atomic decomposition of $L^2(\mathbb{R})$ with respect to $\ell^2(\mathbb{Z}^2)$.

5 Wavelet expansion

The theory of wavelets is based on the notion that a function can be constructed from translated and dilated versions of a generator. Such a generator ψ is in general a square-integrable function and is called *admissible* if it satisfies the *Calderón admissible condition.*

Definition 9. Let $\psi\in L^2(\mathbb{R})$. The admissible constant associated with ψ is defined as

$$C_\psi := \int_{\mathbb{R}\setminus\{0\}} \frac{\left|\hat{\psi}\,(\xi)\right|^2}{|\xi|} d\xi, \tag{25}$$

where $\hat{\psi}(\xi)$ denotes the Fourier transform of $\psi(t)$. If $C_\psi\in\mathbb{R}^+$, then ψ is called *admissible.*

Remark 3. Another form to express the admissibility of $\psi\in L^2(\mathbb{R})$ is that it should satisfy

$$\frac{\hat{\psi}(\xi)}{\sqrt{\xi}} \in L^2(\mathbb{R}). \tag{26}$$

Translated and dilated versions of a generator $\psi\in L^2(\mathbb{R})$ are obtained by applying the translation operator and dilation operator on ψ.

Definition 10. Let $\psi\in L^2(\mathbb{R})$. The dilation operator $\mathcal{D}_v$, that dilates a function by $v\in\mathbb{R}\backslash\{0\}$, is given by

$$\mathcal{D}_v : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (\mathcal{D}_v\psi)(t) = \frac{1}{\sqrt{|v|}}\psi\left(\frac{t}{v}\right), \quad v\in\mathbb{R}\setminus\{0\}.$$

The commutation relation of the translation and dilation operator is given in the next lemma.

Lemma 4. *For all $\tau\in\mathbb{R}$ and $v\in\mathbb{R}\backslash\{0\}$ the following hold:*

$$(\mathcal{T}_\tau\mathcal{D}_v\psi)(t) = (\mathcal{D}_v\mathcal{T}_{\tau/v}\psi)(t).$$

Proof. The relation follows from the direct computation,

$$\begin{aligned}(\mathcal{T}_\tau\mathcal{D}_v\psi)(t) &= \mathcal{T}_\tau\left(\frac{1}{\sqrt{|v|}}\psi\left(\frac{t}{v}\right)\right),\\ &= \frac{1}{\sqrt{|v|}}\psi\left(\frac{t}{v}-\frac{\tau}{v}\right),\\ &= (\mathcal{D}_v\mathcal{T}_{\tau/v}\psi)(t).\end{aligned}$$

□

A version of a translated and dilated generator ψ is

$$(\mathcal{T}_\tau \mathcal{D}_v \psi)(t) = \frac{1}{\sqrt{|v|}} \psi \left(\frac{t-\tau}{v} \right). \tag{27}$$

A discretized version of such a translated and dilated generator can be obtained by setting $v = v^j$, $\tau = n\alpha v^j$ with $j, n \in \mathbb{Z}$ and $v > 1, \alpha > 0$. Such a discretized version of a translated and dilated wavelet is called a *wavelet atom*

Definition 11. Let $\psi \in L^2(\mathbb{R})$ and $v > 1, \alpha > 0$. The collection of functions given by

$$\mathcal{W}(\psi, v, \alpha) = \{\psi_{j,n}\}_{j,n\in\mathbb{Z}}, \quad \psi_{j,n} := \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi.$$

Definition 12. A wavelet system $\mathcal{W}(\psi, v, \alpha)$ that satisfies the frame condition (3) is called a *wavelet frame.*

There are several sufficient conditions for the system $\{\mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi\}_{j,n\in\mathbb{Z}}$ to form a wavelet frame for $L^2(\mathbb{R})$. One of these sufficient conditions is stated in [1].

Theorem 9. *Let $\psi \in L^2(\mathbb{R})$ and $\alpha > 0, \beta > 1$. Suppose that*

$$A := \inf_{|\xi|\in[1,v]} \left(\sum_{j\in\mathbb{Z}} |\hat{\psi}(v^j \xi)|^2 - \sum_{n\neq 0} \sum_{j\in\mathbb{Z}} |\hat{\psi}(v^j\xi)\hat{\psi}(v^j\xi + \frac{n}{\alpha})| \right) > 0,$$

$$B := \sup_{|\xi|\in[1,v]} \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |\hat{\psi}(v^j\xi)\hat{\psi}(v^j\xi + \frac{n}{\alpha})| < \infty.$$

Then $\{\mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi\}_{j,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ with frame bounds $\frac{A}{\alpha}$ and $\frac{B}{\alpha}$.

Proof. See [2]. □

5.1 Operators associated with wavelet frames

In this section the wavelet system $\mathcal{W}(\psi, v, \alpha)$ is assumed to form a wavelet frame for $L^2(\mathbb{R})$.

The coefficient operator $\mathcal{C}$ associated with $\mathcal{W}(\psi, v, \alpha)$ is given by

$$\mathcal{W} : L^2(\mathbb{R}) \to \ell^2(\mathbb{Z}^2), \quad \mathcal{W}f = \{\langle f, \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi \rangle\}_{j,n\in\mathbb{Z}} \tag{28}$$

This coefficient operator maps a function f into a collection of scalars $\{\langle f, \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi\rangle\}_{j,n\in\mathbb{Z}}$ in the so-called *time-scale plane.* This collection of scalars are called the wavelet coefficients associated with $\mathcal{W}(\psi, v, \alpha)$ and are points on the lattice $\Gamma := \alpha v^j \mathbb{Z} \times v^j \mathbb{Z}$.

The adjoint of the coefficient operator $\mathcal{C}$ associated with $\mathcal{W}(\psi, v, \alpha)$ is the reconstruction operator $\mathcal{R}$ associated with $\mathcal{W}(\psi, v, \alpha)$. This reconstruction operator is given by

$$\mathcal{R} : \ell^2(\mathbb{Z}^2) \to L^2(\mathbb{R}), \quad \mathcal{R}c = \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} c_{j,n} \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi. \tag{29}$$

This synthesis operator constructs a function f from a collection of scalars in the time-scale plane. The concatenation of $\mathcal{C}$ and $\mathcal{R}$, that is, $\mathcal{RC}$, is the frame operator $\mathcal{S}$ associated with $\mathcal{W}(\psi, v, \alpha)$. This frame operator is explicitly given by

$$\mathcal{S} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathcal{S}f = \sum_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \langle f, \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi \rangle \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi. \tag{30}$$

In general, the frame operator associated with a wavelet frame, and its inverse, doesn't commute with the translation operator $\mathcal{T}_{n\alpha v^j}$. For this reason the next inequality holds in general,

$$\mathcal{S}^{-1} \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \psi \neq \mathcal{T}_{n\alpha v^j} \mathcal{D}_{v^j} \mathcal{S}^{-1} \psi. \tag{31}$$

This inequality shows that the canonical dual of a wavelet frame doesn't need to be a wavelet frame itself.

As opposed to the translation operator $\mathcal{T}_{n\alpha v^j}$, the frame operator associated with a wavelet frame does commute with the dilation operator $\mathcal{D}_{v^j}$.

Lemma 5. *Let $\mathcal{D}_{\upsilon^j}$ be the dilation operator and $\mathcal{S}$ be the frame operator associated with a wavelet frame, then*

(i)

$$\mathcal{S}\mathcal{D}_{\upsilon^j} = \mathcal{D}_{\upsilon^j}\mathcal{S}, \quad \forall j \in \mathbb{Z}.$$

(ii)

$$\mathcal{S}^{-1}\mathcal{D}_{\upsilon^j} = \mathcal{D}_{\upsilon^j}\mathcal{S}^{-1}, \quad \forall j \in \mathbb{Z}.$$

Proof. (i) First, note that for a wavelet frame for $L^2(\mathbb{R})$ it holds that

$$\mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi = \mathcal{D}_{\upsilon^j}\mathcal{T}_{n\alpha}\psi.$$

Next, it can be deduced from a direct computation that for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned}
\mathcal{S}\mathcal{D}_{\upsilon^j} f &= \sum_{j' \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle \mathcal{D}_{\upsilon^j} f, \mathcal{D}_{\upsilon^{j'}}\mathcal{T}_{n\alpha}\psi\rangle \mathcal{D}_{\upsilon^{j'}}\mathcal{T}_{n\alpha}\psi, \\
&= \sum_{j' \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \mathcal{D}_{\upsilon^{j'-j}}\mathcal{T}_{n\alpha}\psi\rangle \mathcal{D}_{\upsilon^{j'}}\mathcal{T}_{n\alpha}\psi, \\
&= \sum_{j' \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \mathcal{D}_{\upsilon^{j'}}\mathcal{T}_{n\alpha}\psi\rangle \mathcal{D}_{\upsilon^{j+j'}}\mathcal{T}_{n\alpha}\psi, \\
&= \mathcal{D}_{\upsilon^j}\mathcal{S}f.
\end{aligned}$$

Here the second equality followed from the alternative form of the wavelet coefficients, namely

$$\begin{aligned}
\langle f, \mathcal{D}_{\upsilon^j}\mathcal{T}_{n\alpha}\psi\rangle &= \frac{1}{\upsilon^{j/2}} \int_{\mathbb{R}} f(t)\overline{\psi(\frac{t}{\upsilon^j} - n\alpha)}dt \\
&= \upsilon^{j/2} \int_{\mathbb{R}} f(\upsilon^j t)\overline{\psi(t - n\alpha)}dt \\
&= \langle \mathcal{D}_{\upsilon^{-j}} f, \mathcal{T}_{n\alpha}\psi\rangle.
\end{aligned}$$

(ii) Since $\mathcal{S}$ is a homeomorphishm, it can be replaced by $\mathcal{S}^{-1}$ in (i).

□

Theorem 10. *If a wavelet system $\{\mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi\}_{j,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$, then there exists a frame $\{\mathcal{D}_{\upsilon^j}\mathcal{S}^{-1}\mathcal{T}_{n\alpha}\psi\}_{j,n\in\mathbb{Z}}$ such that any $f \in L^2(\mathbb{R})$ can be represented as the series*

$$f = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi\rangle \mathcal{D}_{\upsilon^j}\mathcal{S}^{-1}\mathcal{T}_{n\alpha}\psi \tag{32}$$

$$f = \sum_{j \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \langle f, \mathcal{D}_{\upsilon^j}\mathcal{S}^{-1}\mathcal{T}_{n\alpha}\psi\rangle \mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi \tag{33}$$

where both series converges unconditionally in the L^2-norm.

Proof. Since, by assumption, the system $\{\mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi\}_{j,n\in\mathbb{Z}}$ forms a frame for $L^2(\mathbb{R})$ its canonical dual is given by $\{\mathcal{S}^{-1}\mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi\}_{j,n\in\mathbb{Z}}$. Since $\{\mathcal{T}_{n\alpha\upsilon^j}\mathcal{D}_{\upsilon^j}\psi\}_{j,n\in\mathbb{Z}}$ is equal to $\{\mathcal{D}_{\upsilon^j}\mathcal{T}_{n\alpha}\psi\}_{j,n\in\mathbb{Z}}$ it follows from lemma 5 that the canonical dual has the form $\{\mathcal{D}_{\upsilon^j}\mathcal{S}^{-1}\mathcal{T}_{n\alpha}\psi\}_{j,n\in\mathbb{Z}}$. Since both (32) and (33) are frame decompositions their unconditional convergence follows from theorem 1. □

This theorem provides two unconditional atomic decompositions of $L^2(\mathbb{R})$ with respect to $\ell^2(\mathbb{Z}^2)$, namely the ones given by a wavelet frame $\mathcal{W}(\psi, \upsilon, \alpha)$ and its canonical dual. Note that the canonical dual frame is not generator from one single generator ψ, but from a collection of generators $\tilde{\psi}_{0,n} = \mathcal{S}^{-1}\mathcal{T}_{n\alpha}\psi$. In general, any dual frame of $\mathcal{W}(\psi, \upsilon, \alpha)$ forms together with this wavelet frame an atomic decomposition.

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