A primer on the theory of frames

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Abstract

This report aims to give an overview of frame theory in order to gain insight in the use of the frame framework as a unifying layer in the Large Time-Frequency Analysis Toolbox (LTFAT). In the first section the general concept of a frame is introduced and some of its properties are defined. In the second section some linear operators associated with frames are introduced and their use for the construction of canonical frames and the computation of the frame bounds are stated. In these first sections frames are notated as general collections or families and the operators associated with frames as mappings since this is also the way they are presented in the LTFAT. In the third section frames and their associated operators are described in terms of matrices in order to gain insight in the computational aspects of frames.

The concept of a frame was first introduced in [5] in the context of nonharmonic Fourier series. In this context the frame forms a system of complex exponentials. Frames were studied in the same context further in [9]. In [4] the concept of a frame was generalized to other contexts than nonharmonic Fourier series which leads to general nonorthogonal expansions in a Hilbert space. In the meantime frame theory became a relevant subject in several disciplines and is studied in several contexts.

1 Frames

A frame of a Hilbert space $\mathcal{H}$ is a collection of elements $\{f_k\}_{k\in I\subseteq\mathbb{Z}} \subseteq \mathcal{H}$ that spans $\mathcal{H}$. Formally, a collection of elements $\{f_k\}_{k\in I} \subseteq \mathcal{H}$ is a frame for $\mathcal{H}$ if there exist two constants $0 < A \leq B < \infty$ for which

$$\forall f \in \mathcal{H}, \quad A\,||f||^2 \;\leq\; \sum_{k\,\in\, I} |\,\langle f, f_k\rangle\,|^2 \;\leq\; B\,||f||^2 \tag{1}$$

The statement expressed in (1) is in general called the frame condition and it guarantees that any $f \in \mathcal{H}$ can be constructed from the frame $\{f_k\}_{k\in I}$.

1.1 Frame bounds

The constants A and B in (3) are called the lower and upper frame bound, respectively. The largest lower frame bound and the smallest upper frame bound are called the optimal frame bounds. The optimal frame bounds A and B are defined as

$$A = \inf_{f \in \mathcal{H}} \frac{\sum_{k \in I} \mid \langle f, f_k \rangle \mid^2}{||f||^2} \tag{2}$$

$$B = \sup_{f \in \mathcal{H}} \frac{\sum_{k \in I} \mid \langle f, f_k \rangle \mid^2}{||f||^2} \tag{3}$$

The frame bounds are intimately related to the reconstruction of f [6]. First, the operator mapping $f \in \mathcal{H}$ into its transform coefficients $|\langle f, f_k \rangle|$ has to be bounded, that is, $\sum_{k \in I} \mid \langle f, f_k \rangle \mid^2 < \infty$, achieved by the upper frame bound B. Second, no f with $||f|| > 0$ should be mapped to 0, achieved by the lower frame bound A.

If $A \to B$, then

$$||f - \tilde{f}|| \to 0 \tag{4}$$

where

$$\tilde{f} \;=\; \frac{2}{A+B} \sum_{k \in I} \langle f, f_k \rangle f_k \tag{5}$$

If the frame bounds A and B are equal the frame is called tight and every $f \in \mathcal{H}$ can be perfectly reconstructed as

$$f = \frac{1}{A} \sum_{k \in I} \langle f, f_k \rangle f_k \tag{6}$$

The tight frame with frame bounds $A = B = 1$ is called a Parseval frame since the frame condition expressed in (1) becomes Parseval's identity:

$$\forall f \in \mathcal{H}, \quad ||f|| = \sum_{k \in I} |\langle f, f_k \rangle| \tag{7}$$

Every $f \in \mathcal{H}$ can be reconstructed from a Parseval frame as

$$f = \sum_{k \in I} \langle f, f_k \rangle f_k \tag{8}$$

The reconstruction formula of a Parseval frame is equal to the reconstruction formula of an orthonormal basis. From this it can be deduced that any orthonormal basis is a Parseval frame.

1.2 Redundancy of a frame

If the frame $\{f_k\}_{k\in I}$ contains more elements than is needed to span $\mathcal{H}$, then the frame is called overcomplete or redundant. A frame $\{f_k\}_{k\in I}$ is called exact if it ceases to be a frame when any single element is deleted from $\{f_k\}_{k\in I}$. An exact frame is equal to a basis since it is a linearly independent collection and the removal of any of its elements results in span$\{f_k\} \neq \mathcal{H}$.

For a a finite-dimensional Hilbert space $\mathcal{H}$ with dimension N this means that the finite frame $\{f_k\}_{k=1}^{L}$ with $L \in \mathbb{N}$ contains more than N elements, that is, $L > N$. The redundancy of a finite frame is defined as the ratio $\frac{L}{N}$. If a finite frame is overcomplete or redundant, that is, $\frac{L}{N} > 1$, the collection $\{f_k\}_{k=1}^{L}$ is linearly dependent. If $\frac{L}{N} = 1$, then the frame is exact and the collection $\{f_k\}_{k=1}^{L}$ is linearly independent. For the redundancy of an infinite frame a so-called redundancy function can be defined, see [1].

2 Operators associated with frames

There are a few important operators associated with frames, including the analysis operator, the synthesis operator, the frame operator and the frame correlation operator . The analysis operator $\mathcal{A} : \mathcal{H} \to \ell^2(I)$ with analysis frame $\{f_k\}$ is given by

$$\mathcal{A}f = \{\langle f, f_k\rangle\}, \qquad k \in I \tag{9}$$

The analysis operator $\mathcal{A}$ maps f to a collection of scalars $\{\langle f, f_k\rangle\}$ called the frame coefficients. The frame coefficients are associated with the analysis frame $\{f_k\}$ and doesn't have to be unique if the frame is redundant.

The synthesis operator $\mathcal{A}^* : \ell^2(I) \to \mathcal{H}$ with synthesis frame $\{f_k\}$ is given by

$$\mathcal{A}^* c = \sum_{k\,\in\, I} c_k f_k \tag{10}$$

The synthesis operator $\mathcal{A}^*$ constructs f as a linear combination of the frame coefficients $\{c_k\}$ and synthesis frame $\{f_k\}$

The concatenation of the synthesis operator $\mathcal{A}^*$ and the analysis operator $\mathcal{A}$, that is, $\mathcal{A}^*\mathcal{A}$, is the so-called frame operator $\mathcal{S} : \mathcal{H} \to \mathcal{H}$ given by

$$\mathcal{S}f = \mathcal{A}^*\mathcal{A}f = \sum_{k\,\in\, I} \langle f, f_k\rangle f_k \tag{11}$$

Since $\langle \mathcal{S}f, f\rangle = \sum_{k\,\in\, J} |\langle f, f_k\rangle|^2$, the frame condition expressed in (1) could also be written in terms of the frame operator $\mathcal{S}$ as

$$\forall f \in \mathcal{H}, \quad A\,||f||^2 \;\leq \langle \mathcal{S}f, f\rangle \leq\; B\,||f||^2 \tag{12}$$

The concatenation of the analysis operator $\mathcal{A}$ and the analysis operator $\mathcal{A}^*$, that is, $\mathcal{A}\mathcal{A}^*$, is the so-called frame correlation operator $\mathcal{R} : \ell^2(I) \to \ell^2(I)$ given by

$$\mathcal{R}c = \mathcal{A}\mathcal{A}^* c = \sum_{n \in I} c_n \langle f_n, f_k \rangle \tag{13}$$

The frame correlation operator $\mathcal{R}$ shares many properties with the frame operator $\mathcal{S}$, but differs mainly from $\mathcal{S}$ in that the range and domain of $\mathcal{R}$ are contained in $\ell^2(I)$, whereas the range of $\mathcal{S}$ is $\mathcal{H}$ [7].

The optimal frame bounds A and B can be computed from both the analysis operator $\mathcal{A}$ and the frame correlation operator $\mathcal{R}$ and its pseudo-inverse $\mathcal{R}^\dagger$ as

$$A = \left|\left|\mathcal{A}^{-1}\right|\right|^{-2} = \left|\left|\mathcal{R}^\dagger\right|\right|^{-1} \tag{14}$$

$$B = \left|\left|\mathcal{A}\right|\right|^{2} = \left|\left|\mathcal{R}\right|\right|^{-1} \tag{15}$$

2.1 The canonical dual and tight frame

The reconstruction of $f \in \mathcal{H}$ from a frame $\{f_k\}$ through the reconstruction formula expressed in (5) leads to a reconstruction error if the frame $\{f_k\}$ is not a tight frame. It is in general possible to reconstruct a $f \in \mathcal{H}$ from a frame $\{f_k\}$ after a so-called dual $\{g_k\}$ is defined. A collection $\{g_k\}$ from which any $f \in \mathcal{H}$ can be represented as

$$f = \sum_{n \, \in \, I} \langle f, g_k \rangle f_k \tag{16}$$

is in general called a dual of $\{f_k\}$. If $\{g_k\}$ satisfies the frame condition expressed in (1), then $\{g_k\}$ is called a dual frame of $\{f_k\}$.

A special type of dual frame is the so-called canonical dual frame. The canonical dual frame $\{\tilde{f}_k\}$ of $\{f_k\}$ is defined as $\{\tilde{f}_k\} = \{\mathcal{S}^{-1} f_k\}$. Any $f \in \mathcal{H}$ can be reconstructed from $\{f_k\}$ and $\{\tilde{f}_k\}$ as

$$f = \sum_{n \, \in \, I} \langle f, \tilde{f}_k \rangle f_k = \sum_{n \, \in \, I} \langle f, f_k \rangle \tilde{f}_k \tag{17}$$

It can be proved that if $\{f_k\}$ is a frame with frame bounds A and B, the canonical dual frame is a frame with frame bounds B^{-1} and A^{-1} [2], that is, $\{\mathcal{S}^{-1} f_k\}$ satisfies

$$\forall f \in \mathcal{H}, \quad B^{-1} \left|\left|f\right|\right|^2 \ \le \ \sum_{k \, \in \, I} \left| \ \langle f, \mathcal{S}^{-1} f_k \rangle \ \right|^2 \ \le \ A^{-1} \left|\left|f\right|\right|^2 \tag{18}$$

If a dual frame $\{g_k\}$ of $\{f_k\}$ is not a canonical dual frame $\{\mathcal{S}^{-1} f_k\}$, that is, if $\{g_k\} \neq \{\mathcal{S}^{-1} f_k\}$, then $\{g_k\}$ is called the alternative dual frame of $\{f_k\}$.

Aside the canonical dual frame $\{\mathcal{S}^{-1}f_k\}$ of $\{f_k\}$ another frame that can be constructed through the frame operator $\mathcal{S}$ is the so-called canonical tight frame. The canonical tight frame of $\{f_k\}$ is defined as $\{\mathcal{S}^{-\frac{1}{2}}f_k\}$ [8]. The canonical tight frame is a tight frame that is related to $\{f_k\}$ and leads to perfect reconstruction if it is used for both analysis and synthesis.

3 Frames and matrices

In this section the Hilbert space $\mathcal{H}$ that will be considered is a finite-dimensional inner product space with dimension n. The finite frame that will be considered is $\{f_k\}_{k=1}^m$. Since m vectors can at most span a m-dimensional space, $m \geq n$ if $\{f_k\}_{k=1}^m$ is a frame for $\mathcal{H}$ [3].

A frame $\{f_k\}_{k=1}^m$ can be represented as a $n \times m$ matrix F by

$$F = \begin{pmatrix} | & | & \cdots & | \\ f_1 & f_2 & \cdots & f_m \\ | & | & \cdots & | \end{pmatrix} \tag{19}$$

In this case is each element f_k of $\{f_k\}_{k=1}^m$ a column vector of F. The matrix F is called a frame matrix. If F is the frame matrix of a tight frame, then $FF^* = \lambda I, \lambda > 0$, and F is called a tight frame matrix. If $FF^* = I$, the frame $\{f_k\}$ is a Parseval frame and F is called the Parseval frame matrix.

The frame condition expressed in (1) can be expressed in terms of the frame matrix F as

$$\forall f \in \mathcal{H}, \quad A\,||f||^2 \;\leq\; ||F^*f||^2 \;\leq\; B\,||f||^2 \tag{20}$$

Here the columns of the matrix F^* form a frame for $\mathcal{H}$ if (20) is satisfied.

Frame bounds A and B of the frame $\{f_k\}_{k=1}^m$ can be computed from F^* as

$$A = \frac{1}{||(F^*)^{-1}||^2} \tag{21}$$

$$B = ||F^*||^2 \tag{22}$$

The redundancy of the frame $\{f_k\}_{k=1}^m$ for the Hilbert space $\mathcal{H}$ is the ratio $\frac{m}{n}$. In section 1.2 a basis was defined as an exact frame, that is, a frame with redundancy 1. From this it can be deduced that a basis corresponds to a square matrix since the redundancy of a frame is only equal to 1 if and only if $m = n$.

3.1 Operators and frame matrices

The frame matrix F expressed in (19) describes the synthesis operator $\mathcal{A}^*$ that was defined in (10). For the frame $\{f_k\}_{k=1}^m$ of $\mathcal{H}$ the synthesis operator $\mathcal{A}^* = F : \mathbb{C}^m \to \mathcal{H}$.

The matrix F^* given by

$$F^* = \begin{pmatrix} - & \bar{f}_1 & - \\ - & \bar{f}_2 & - \\ \vdots & \vdots & \vdots \\ - & \bar{f}_m & - \end{pmatrix} \tag{23}$$

describes the analysis operator $\mathcal{A}$ defined in (9) as $\mathcal{A} = F^* : \mathcal{H} \to \mathbb{C}^m$.

The frame operator $\mathcal{S}$ is defined in terms of matrices as $\mathcal{S} = FF^* : \mathcal{H} \to \mathcal{H}$. The optimal frame bounds A and B correspond to the smallest and largest eigenvalues of the matrix FF^*, respectively [3]. The frame correlation operator $\mathcal{R}$ is defined in terms of matrices as $\mathcal{R} = F^*F = \mathbb{C}^m \to \mathbb{C}^m$.

Since the canonical dual frame $\{\tilde{f}_k\}_{k=1}^m$ of $\{f_k\}_{k=1}^m$ can be computed from the frame operator $\mathcal{S} = FF^*$, it can be represented as a matrix. The elements $\tilde{f}_k$ of the canonical dual frame $\{\tilde{f}_k\}_{k=1}^m = \{\mathcal{S}^{-1}f_k\}_{k=1}^m$ of $\{f_k\}_{k=1}^m$ are represented as the columns of the matrix $(FF^*)^{-1}F$. In the same way the canonical tight frame of $\{f_k\}_{k=1}^m$ can be expressed as a matrix as $(FF^*)^{-\frac{1}{2}}F$.

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