Designing tight filter bank frames for nonlinear frequency scales

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***Abstract*—We propose a method for the Fourier-side design of nonuniform filter banks. Given a frequency scale and a frequency response prototype, we obtain a family of filters that are uniform shape when observed on the given scale. We provide necessary and sufficient conditions for the resulting (analysis) filter bank to form a (tight) frame. Implementation of the filter bank analysis and reconstruction are discussed and complemented with a number of examples.**

I. Introduction

Herein, we introduce a novel family of filter banks (FBs) adapted to nonlinear frequency scales. Uniquely determined by the choice of a single prototype filter and a warping function that determines the desired frequency scale.

The desire for FBs providing adapted time-frequency resolution has sparked a wealth of research and the construction of various systems with vastly different properties. The most prominent such systems are those in the extended *wavelet* [1] family, all of which are adapted to a logarithmic frequency scale, constant-Q FBs [2]–[4] which are in fact wavelet FBs in disguise and Gammatone FBs [5], [6]. Other important examples include discrete variants of the α-transform [7], [8], a parametrized family of FBs adapted to frequency scales between linear and logarithmic.

The idea of warping of the frequency axis to obtain adapted filter banks is not entirely new and was already used in the proof of the so called painless conditions for wavelets [9]. A number of other methods for obtaining *warped filter banks* have been proposed, e.g. by applying a unitary basis transformation to Gabor or wavelet atoms [10]–[13]. Although unitary transformation bequeaths basis (or frame) properties to the warped atoms, the resulting system is not anymore a filter bank. Instead, the warped system provides an undesirable, irregular time-frequency tiling, see [12].

Closer to our own approach, Braccini and Oppenheim [14], as well as Twaroch and Hlawatsch [15], propose a warping of the filter frequency responses only, by defining a unitary warping operator. However, in ensuring unitarity, the authors give up the property that warping is shape preserving when observed on the warped frequency scale. In this contribution, we trade the unitary operator for a shape preserving warping in order to construct tight and well-conditioned frames more easily.

II. Notation and preliminaries

In the following, we consider signals in $\ell_2(\mathbb{Z})$ sampled at the frequency ξ_s. We write $\hat{f}(\xi) := \mathcal{F}f(\xi) = \sum_{\mathbb{Z}} f[l]e^{-2\pi il\xi}$, for the Fourier transform on $\ell^1(\mathbb{Z})$ and its extension to $\ell^2(\mathbb{Z})$, denoting its inverse by $\check{f} := \mathcal{F}^{-1}f$. Further, we require the *translation operator* defined by $\mathbf{T}_k f = f[\cdot - k]$. The inner product of two signals x, y is $\langle x, y\rangle = \sum_n x[n] \cdot \overline{y[n]}$ and by $\|\cdot\| := \|\cdot\|_2$, we denote the natural norms on $\ell^2(\mathbb{Z})$ and $\mathbf{L}^2(\mathbb{T})$, respectively.

A *($K+1$ channel, nonuniform analysis) filter bank* (FB) can be considered as a collection $\{g_{n,k}\}_{n,k}$ of $K+1$ shift-invariant systems

$$g_{n,k}[l] := \mathbf{T}_{na_k}\check{g}_k[l], \quad a_k \in \mathbb{N}, \tag{1}$$

with *filters* $\check{g}_k$ and *downsampling factors* a_k. In this contribution, we consider $\check{g}_k \in \ell^2(\mathbb{Z})$ only. The Fourier transform $\hat{g}$ of a filter g is called *frequency response*. A FB forms a *frame*, if there are positive constants A, B such that

$$A\|f\|^2 \leqslant \sum_{k=0}^{K}\sum_{n\in\mathbb{Z}} |c_{n,k}|^2 \leqslant B\|f\|^2, \ \forall \ f \in \ell^2(\mathbb{Z}), \tag{2}$$

where $c_{n,m} = \langle f, g_{n,k}\rangle$ are the FB coefficients. $\{g_{n,k}\}_{n,k}$ is a *tight frame*, if $A = B$. The frame property ensures perfect reconstruction from the FB coefficients by means of a *dual frame* $\{\widetilde{g_{n,k}}\}_{n,k}$ and the formula

$$f[l] = \sum_{k=0}^{K}\sum_{n\in\mathbb{Z}} c_{n,k}\widetilde{g_{n,k}}[l]. \tag{3}$$

Note that, for arbitrary downsampling factors a_k, we cannot guarantee that there is a dual frame that is also a $K+1$ channel FB with downsampling factors a_k.

III. Warped filter banks

We call a continuous, increasing function $F : D \to \mathbb{R}$, where $D \in \{\mathbb{R}, \mathbb{R}^+\}$, a *warping function*. to be odd for convenience. A warping function determines a mapping from the linear frequency axis (measured in Hz) onto some nonlinear frequency scale and the composition $\theta \circ F$ is a function that has the shape of θ, when observed on the new scale. Hence, the system

$$\{\theta_{F,k}\}_{k\in\mathbb{Z}}, \text{ with } \theta_{F,k} := ((\mathbf{T}_k\theta) \circ F). \tag{4}$$

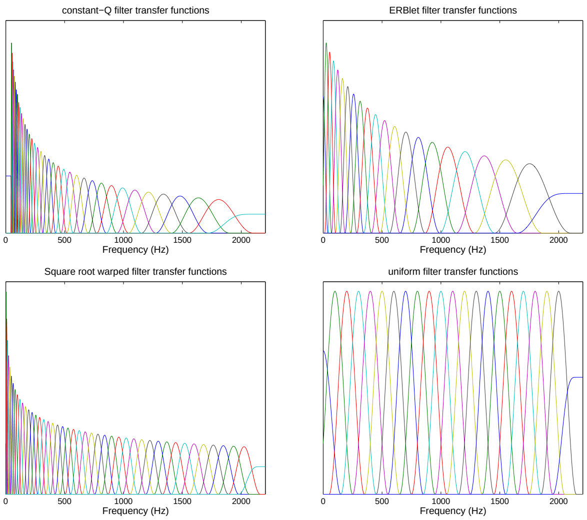

Fig. 1. Frequency responses of warped filters using a Hann window prototype: (top-left) logarithmic warping $F(x) = 10\log(x)$, (top-right) ERBlet warping $F(x) = 21.4\,\mathrm{sgn}(x)\log_{10}(1 + |x|/229)$, (bottom-left) square root warping $F(x) = \mathrm{sgn}(x)(\sqrt{1 + |x|} - 1)$ and (bottom-right) linear warping $F(x) = x/100$. The systems use 1 bin/unit and were restricted to the frequency range 0 Hz-8.82 kHz for visualization.

clearly provides a sort of shift-invariant system on the scale described by F, see Figure 1.

Warped FBs are constructed by taking (a suitable subset of) $\{\theta_{F,k}\}_{k\in\mathbb{Z}}$ as frequency responses g_k, i.e. we define $g_k := \sqrt{a_k}\theta_{F,k}$, where the role of the normalization factor is clarified in the next sections. The main task is the restriction of the system onto the frequency range determined by the sampling rate of ξ_s Hz, i.e. $]-\xi_s/2, \xi_s/2] \subseteq \mathbb{R}$ or $]0, \xi_s/2] \subseteq \mathbb{R}^+$. Although there are various ways of performing this restriction, they should provide similar results for all reasonable signals. Hence, we propose here a straightforward approach.

First, choose a continuous function θ with short support $[c, d] := \mathrm{supp}(\theta)$, contained in $F^{-1}(I_{fr})$. Here $I_{fr} = [-\xi_s/2, \xi_s/2]$, resp. $I_{fr} =]0, \xi_s/2]$. In the first case, we define

$$k_{\max} = \max\{k \in \mathbb{Z} \ : \ F^{-1}(k + d) \leqslant \xi_s/2\}$$
$$k_{\min} = \min\{k \in \mathbb{Z} \ : \ F^{-1}(k + c) > -\xi_s/2\}$$

and obtain the frequency responses

$$g_k(\xi/\xi_s) := \sqrt{a_k}\theta_{F,k}(\xi), \ \xi \in I_{fr}, \tag{5}$$

for all $k \in [k_{\min}, k_{\max}]$ and

$$g_{k_{\max}+1}(\xi/\xi_s) := \left(a_{k_{\max}+1} \sum_{k\in\mathbb{Z}\setminus[k_{\min},k_{\max}]} |\theta_{F,k}(\xi)|^2 \right)^{1/2}, \tag{6}$$

$\xi \in I_{fr}$.

In the second case, i.e. $I_{fr} =]0, \xi_s/2]$, it would be theoretically sufficient to copy the scheme above and set $k_{\min} = -\infty$. To obtain a FB with a finite number of filters, select $k_{\min}$ and use instead

$$\begin{aligned} g_{k_{\max}+1}(\xi/\xi_s) &:= \left(a_{k_{\max}+1} \sum_{k>k_{\max}} |\theta_{F,k}(\xi)|^2 \right)^{1/2}, \\ g_{k_{\min}}(\xi/\xi_s) &:= \left(a_{k_{\min}} \sum_{k\leqslant k_{\min}} |\theta_{F,k}(\xi)|^2 \right)^{1/2}, \end{aligned} \tag{7}$$

$\xi \in I_{fr}$.

We denote a warped FB by $\mathcal{G}(\theta, F, \mathbf{a}) := \{g_{n,k}\}_{n,k}$, where $\mathbf{a} := \{a_k\}_{k\in I_K}$, $g_{n,k} = \mathbf{T}_{na_k}\mathcal{F}^{-1}g_k$ for all $k \in I_K$ and $I_K = [k_{\min}, k_{\max} + 1]$. Note that, in the case $D = \mathbb{R}^+$, $\mathcal{G}(\theta, F, \mathbf{a})$ depends on the choice of $k_{\min}$.

The implementation we provide, see Section V, additionally allows for the selection of an integer $bins/unit$ parameter B. For $B \neq 1$, the warped filters are constructed from the translates $\mathbf{T}_{k/B}\theta$ instead. This is equivalent to setting $F_{\text{new}} = B \cdot F$ and $\theta_{\text{new}}(x) = \theta(x/B)$.

Remark 1. The case $I_{fr} =]0, \xi_s/2]$ of analytic FBs is relevant, e.g. for the analysis of real-valued signals such as audio. Selecting an appropriate cut-off $k_{\min} - 1$ is particularly important for implementation, where we in fact consider sequences of finite length L. Here, the frequency resolution is determined by the sampling rate ξ_s and the signal length L. Since an infinite number of filters is placed in any neighborhood of the zero frequency, the discretized filters in the low frequency region do not bear any resemblance to their continuous model and will show completely different time-frequency concentration. Therefore, we propose a single filter, covering all these critical cases and preserving the summation properties again.

In practice, the high- and low-pass filters $g_{k_{\max+1}}$ and $g_{k_{\min}}$ can often be chosen such that the information contained in the corresponding bands is irrelevant to the user.

IV. Warped FB frames

In addition to providing filters with uniform frequency resolution on the desired frequency scale, the warped FB structure allows the derivation of easy-to-satisfy necessary and sufficient frame conditions. The following proposition is a direct consequence of a result for general FB frames, see [16] for analogous results for continuous-time signals.

Proposition 1. *Let θ be such that $\mathrm{supp}(\theta) \subseteq [c, d]$ for some constants $c < d$ and $\mathcal{G}(\theta, F, \mathbf{a})$ a corresponding warped FB as defined by* (5) *and* (6)*, resp.* (5) *and* (7)*. If $\mathcal{G}(\theta, F, \mathbf{a})$ is a frame, then there exist positive constants A, B such that*

$$0 < A \leqslant \sum_{k\in\mathbb{Z}} |\mathbf{T}_k\theta|^2 \leqslant B < \infty. \tag{8}$$

If furthermore, for all $k \in I_K\backslash\{k_{max} + 1\}$, $a_k^{-1} \geqslant F^{-1}(d + k) - F^{-1}(c + k)$ and

$$a_{k_{max}+1}^{-1} \geqslant 1 - F^{-1}(c + k_{max} + 1) + F^{-1}(d + k_{min}),$$

if $D = \mathbb{R}$, respectively

$$\begin{aligned} a_{k_{max}+1}^{-1} &\geqslant 1 - F^{-1}(c + k_{max} + 1), \\ a_{k_{min}}^{-1} &\geqslant F^{-1}(d + k_{min}) \end{aligned}$$

if $D = \mathbb{R}^+$, then $\mathcal{G}(\theta, F, \mathbf{a})$ is a frame with frame bounds A, B if and only if (8) *holds. In that case, the system $\mathcal{G}(\widetilde{\theta}, F, \mathbf{a})$, with*

$$\widetilde{\theta} = \frac{\theta}{\sum_k |\mathbf{T}_k \theta|^2}, \text{ a.e..} \tag{9}$$

is a dual frame for $\mathcal{G}(\theta, F, \mathbf{a})$.

Remark 2. Clearly, this implies that $\sum_{k\in\mathbb{Z}} |\mathbf{T}_k\theta|^2 = \text{const.}$ is necessary for (and sometimes equivalent to) the tight frame property. Such families are easy to construct, though.

Sometimes, higher downsampling rates than the ones allowed by the second part of Prop. 1 are desired. In that case, it becomes harder to determine the frame property and unclear whether there is a dual frame of the form $\mathcal{G}(\widetilde{\theta}, F, \mathbf{a})$ for some function $\widetilde{\theta}$.

For any (general) filter bank, it is a consequence of the FB structure that it forms a frame if

$$\begin{aligned} &\sum_{k\in I_K} a_k^{-1} |g_k|^2(\xi) \\ &> \sum_{k\in I_K} \left(a_k^{-1} |g_k| \sum_{n=1}^{a_k - 1} |\mathbf{T}_{n a_k^{-1}} \overline{g_k}|(\xi) \right) =: \mathcal{A}(\xi), \end{aligned} \tag{10}$$

for all $\xi \in I_{fr}/\xi_s$ as this implies invertibility of the frame operator. In signal processing terms, the equation above means that the combined aliasing components (using the analysis FB for synthesis also) are smaller in magnitude than the main component.

For a warped FB $\mathcal{G}(\theta, F, \mathbf{a})$, the condition $a_k^{-1} \geqslant F^{-1}(d + k) - F^{-1}(c+k)$ implies that the right hand side of (10) equals 0. Consequently, if we assume the warping function F and prototype θ to be smooth, in particular if θ is nonincreasing away from some central point, then we can increase the downsampling rates a_k to some degree beyond the bounds given in Prop. 1 without violating $\max_\xi \mathcal{A}(\xi) < A$. Hence, we can reduce the redundancy of the warped FB, often by a surprising amount, without violating the frame property, see Section V. For the situation of continuous-time signals, a more explicit result is presented in [16, Theorem 2].

V. Implementation & Examples

In the previous sections, warped FBs were defined for prototypes θ with small support only, producing a family of bandlimited filters. Although non-bandlimited filters are feasible, provided we employ a suitable treatment of frequency range borders, in general we cannot hope to obtain FIR filters. For finite time signals in $\mathbb{C}^L$, results equivalent to Section IV hold and discrete systems are obtained by sampling the torus $\mathbb{T}$ appropriately, yielding discrete variants of the g_k. Hence, we can employ algorithms based on fast convolution via application of FFT. To analyze signals of arbitrary length or sequences in $\ell^2(\mathbb{Z})$, a suitable blocking scheme, see [4] for a scheme that respects the frame property, can be used.

We use FB algorithms from the open-source MATLAB/octave toolbox LTFAT (‘Large time-frequency analysis toolbox’) [17] (http://ltfat.sourceforge.net/). The methods for warped filter banks are provided in the current development version, available on the webpage above, and scheduled to be included in the next release. In particular, `warpedblfilter` and `warpedfilters` compute warped filters and provide a full warped FB, respectively, when supplied with a prototype function, a warping function and its inverse. Moreover, the method `erbfilters` provides an option for using warped filters instead of the usual symmetric ones, although the border treatment in the latter deviates slightly from what we describe here. Analysis, synthesis and other filter bank operations are provided in the `filterbank`, `blockproc` and `frames` packages of LTFAT. For more information and usage instructions, refer to the LTFAT documentation.

We provide MATLAB scripts that reproduce the figures and tables in this contribution on the webpage http://ltfat.sourceforge.net/notes/039/, where the test signal used in Figure 2 can also be found.

Reconstruction: In general, the existence of a dual FB having the same number of filters $\widetilde{g_k}$ and upsampling factors $\mathbf{a}$ as $\mathcal{G}(\theta, F, \mathbf{a})$ cannot be guaranteed. Therefore, we use three different approaches to compute the action of the synthesis FB:

(i) In the setting of Prop. 1, i.e. for bandlimited filters with sufficiently dense sampling, we use (9) to efficiently compute the dual FB. Synthesis is then accomplished by a standard nonuniform FB synthesis algorithm.

(ii) If the conditions of Prop. 1 are violated but $\sum_{k\in I_k} \operatorname{lcm}(\mathbf{a})/a_k$ is small enough, $\mathcal{G}(\theta, F, \mathbf{a})$ can be transformed into an equivalent uniform FB, i.e. a FB with uniform downsampling factor $\operatorname{lcm}(\mathbf{a})$, (see [18] for details) and a dual FB can be easily obtained using standard algorithms for the computation of dual uniform FBs.

(iii) If the number of channels in the equivalent uniform FB is too large, the computation and storage of the dual FB become unfeasible. In such cases, the action of the canonical dual FB is computed using a conjugate gradients (CG) algorithm. Iterative synthesis via CG benefits from the fact that although the number of iterations necessary to achieve the desired precision depends on the actual frame bound ratio of the analysis FB, it does not require explicit estimates of the frame bounds as opposed to other iterative approaches like the classical frame algorithm [19]. Furthermore, since each iteration computes the analysis followed by synthesis with $\mathcal{G}(\theta, F, \mathbf{a})$, the algorithm's complexity is independent of the structure of the dual FB. Similar to the results in [20] it can be expected that a preconditioner often drastically reduces the number of iterations required to achieve a certain precision.

Examples: We now mention a few interesting examples from the vast selection of possible warping functions.

Example 1 (constant-Q). Choosing $F = \log_b$, with $D = \mathbb{R}^+$ leads to a system of the form

$$\theta_{F,k}(t) = \theta(\log_b(t) - k) = \theta_{F,0}(\log_b(tb^{-k})).$$

Hence we obtain a family of dilates with constant center frequency to bandwidth ratio (Q-factor) and it is natural to choose $\mathbf{a}$ as the sequence of $a_k = \tilde{a} b^{-k}$ for some constant $\tilde{a}$.

Example 2. The family of warping functions $F_l(t) = c\left((t/d)^l - (t/d)^{-l}\right)$, for some $c, d > 0$ and $l \in (0, 1]$, is an alternative to the logarithmic warping for the domain $D = \mathbb{R}^+$. This type of warping provides a frequency scale that approaches the limits 0 and ∞ of the frequency range D in a slower fashion than the constant-Q warping. In other words, $\theta_{F_l,k}(t)$ is less deformed for $k > 0$, but more deformed for $k < 0$ than in the case $F = \log_b$.

Example 3 (ERBlets). In psychoacoustics, the investigation of filter banks adapted to the spectral resolution of the human ear has been subject to a wealth of research, see [21] for an overview. We mention here the Equivalent Rectangular Bandwidth scale (ERB-scale) described in [22], which introduces a set of bandpass filters following the human perception. In [20] the authors construct a filter bank that is tailored to the ERB-scale. A similar FB can be constructed using warped filters. In our terminology the ERB warping function is given by

$$F_{\text{ERB}}(t) = \operatorname{sgn}(t)\, c_1 \log\left(1 + \frac{|t|}{c_2}\right), \tag{11}$$

where the constants are given by $c_1 = 9.265$ and $c_2 = 228.8$.

Example 4. The warping function $F(t) = \operatorname{sgn}(t)\left((|t| + 1)^l - 1\right)$ for some $l \in (0, 1]$ leads to a filter bank that is structurally very similar to a discrete α-*transform*, see [7], [8]. Both of these systems provide a method for constructing FBs adapted to frequency scales ranging from linear ($l = 1$) to *almost logarithmic* ($l \to 0$). In some sense, warping functions of the form (11) are the natural limit for $l \to 0$.

Efficiency and the frame property: Similar to other FBs comprised of bandlimited filters, the analysis and synthesis with a warped filter bank with a fast convolution method has complexity $\mathcal{O}(L \log L)$, see e.g. [23] for a detailed analysis and performance tests. A blocking scheme, such as the one proposed in [4] and implemented in LTFAT, achieves linear complexity with good performance and only minor aliasing effects in the FB coefficients, while preserving perfect reconstruction.

Computing the dual filters given by (9) is a linear cost operation, while computing a dual uniform FB involves the inversion of $L/\operatorname{lcm}(\mathbf{a})$ matrices of size $\operatorname{lcm}(\mathbf{a}) \times \operatorname{lcm}(\mathbf{a})$ resulting in cubic (or slightly better) complexity. But note that dual filters can be precomputed and reused. Iterative inversion via CG requires, for a given warped FB, a constant number of analysis/synthesis steps and is therefore of the same complexity order as the analysis and synthesis algorithms. More details can be found in [20].

Signal analysis:

To demonstrate the effect of different warping functions, we have computed filter bank coefficients of a test signal with the following warping functions:

- $F(x) = 10 \log(x)$, obtaining a constant-Q filter bank,
- $F(x) = 21.4 \operatorname{sgn}(x) \log_{10}(1 + |x|/229)$, obtaining an ERBlet filter bank,
- $F(x) = \operatorname{sgn}(x)(\sqrt{1 + |x|} - 1)$ and

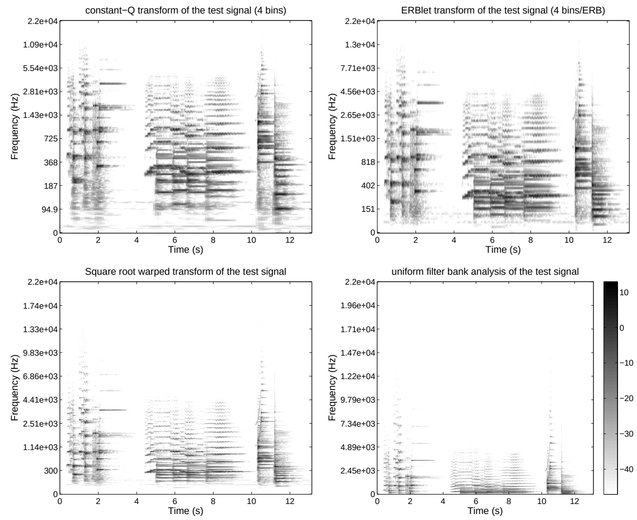

Fig. 2. Time-frequency plots of the "Kafziel" test signal associated with different warping functions, using 4 bins/unit each: (top-left) logarithmic warping $F(x) = 10 \log(x)$, (top-right) ERBlet warping $F(x) = 21.4 \operatorname{sgn}(x) \log_{10}(1 + |x|/229)$, (bottom-left) square root warping $F(x) = \operatorname{sgn}(x)(\sqrt{1 + |x|} - 1)$ and (bottom-right) linear warping $F(x) = x/100$. Intensity is in dB, the colorbar on the bottom-right is valid for all plots.

- $F(x) = x/100$, obtaining a uniform filter bank.

For the logarithmic warping, we selected $k_{\text{min}} = \lfloor F(50) \rfloor$, and 0 for the other examples. The $bins/unit$ parameter was set to $B = 4$. The test signal "Kafziel" is a 12 sec excerpt from a piece for piano and violin sampled at 44.1 kHz. Results are provided in Figure 2, illustrating how different warpings emphasize different frequency regions and provide their own time/frequency resolution trade-off.

Stability of the frame bounds: For the warping functions already used in the previous paragraphs, we have analyzed the dependence of the frame bounds and redundancy. The starting point of our experiment is a Hann window prototype with $2/3$ overlap and maximal downsampling rates satisfying the second part of Prop. 1, yielding (approximately[1]) tight frames with redundancy 3 in all cases.

To obtain lower redundancy systems, we simply divide the downsampling factors by a *redundancy factor*. For reasons of stability, the downsampling factors $a_{k_{\text{min}}}$ and $a_{k_{\text{max}}+1}$ are not modified, but instead the filters $g_{k_{\text{min}}}$ and $g_{k_{\text{max}}+1}$ are appropriately re-normalized.

Table I shows estimates for the frame bound ratio of the resulting analysis FBs. The estimate for the upper frame bound is obtained through the MATLAB internal function `eigs`, supplied with an efficient implementation of the frame operator of the warped FB (analysis, followed by synthesis with $\{g_{n,k}\}_{n,k}$). For any redundancy factor smaller than 1, estimating the lower frame bound additionally requires an inverse of the frame operator, which we obtain by conjugate gradients iterations (`pcg` in MATAB). Therefore, the results

[1] Sampling issues might produce slight deviations if some of the used filters have very small bandwidth.

TABLE I

FRAME BOUND RATIOS OF VARIOUS WARPED FILTER BANKS. FROM TOP TO BOTTOM: LINEAR WARPING $F(x) = x/100$, SQUARE ROOT WARPING $F(x) = \operatorname{sgn}(x)(\sqrt{1+|x|}-1)$, ERBLET WARPING $F(x) = 21.4\operatorname{sgn}(x)\log_{10}(1+|x|/229)$ AND LOGARITHMIC WARPING $F(x) = 10\log(x)$. THE COLUMNS INDICATE THE REDUNDANCY FACTOR, COMPARED TO THE MINIMAL REDUNDANCY SATISFYING PROP. 1.

	1	7/8	3/4	5/8
linear	1.000	1.016	1.256	2.708
square root	1.000	1.080	1.360	3.611
ERB	1.000	1.043	1.481	4.360
logarithmic	1.014	1.080	1.536	4.438

TABLE II

ACTUAL REDUNDANCY OF THE FILTER BANKS USED IN THE FRAME BOUND EXPERIMENT. ROWS AND COLUMNS INDICATE THE WARPING FUNCTION AND REDUNDANCY *factor* RESPECTIVELY, SEE TABLE I.

	1	7/8	3/4	5/8
linear	2.970	2.617	2.255	1.893
square root	2.946	2.594	2.235	1.880
ERB	2.828	2.508	2.185	1.864
logarithmic	2.832	2.511	2.185	1.859

in Table I are subject to numerical inaccuracies and might deviate slightly from the actual frame bound ratio. In Table II, we also provide the actual redundancy of the FBs used in this example.

VI. CONCLUSION

We propose a novel method for the construction of filter banks adapted to nonlinear frequency scales, such that it is straightforward to achieve the tight frame property simply by selecting a good filter prototype and sufficiently small downsampling factors. We showed experimentally that the frame properties are quite robust under higher decimation, achieving low redundancy.

The associated analysis and synthesis operations can be efficiently implemented and reconstruction filters can either be precomputed or simulated through iterative reconstruction. All the necessary methods and some warped families are already implemented in the open-source LTFAT Toolbox (http://ltfat.sourceforge.net/).

A reproducible research addendum with MATLAB scripts that compute the plots and tables in this contribution is available at http://ltfat.sourceforge.net/notes/039/. For deeper results on warped systems for continuous-time signals, refer to [16] and [24]. The warping procedure we propose has already proven useful in the area of graph signal processing [25].

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